CoProD'15: 8th International Workshop on Constraint Programming and Decision Making

CoProD'15: 8th International Workshop on Constraint Programming and Decision Making
November 6th, 2015, at the University of Texas at El Paso
Physical Sciences Building Room 208

8:30 a.m.	**Registration and Light Breakfast**
9:00 a.m.	**Welcome**
9:15 a.m.	**Juan Carlos Figueroa Garcia,** "A Note on the Fuzzy Extension Principle for LP Problems with Fuzzy Coefficient Matrix"
10:00 a.m.	**Martine Ceberio, Olga Kosheleva, and Vladik Kreinovich**, "Constraint Approach to Multi-Objective Optimization"
10:30 a.m.	**Leobardo Valera**, "Using Interval Constraint Solving Techniques to Predict Behavior of Dynamic Systems"
11:00 a.m.	**Break**
11:15 a.m.	**Angel Garcia Contreras**, "Global Optimization via Speculation"
11:45 p.m.	**Martine Ceberio, Olga Kosheleva, and Vladik Kreinovich**, "Optimizing pred(25) Is NP-Hard"
12:15 p.m.	**Lunch on your own**
2:15 p.m.	**Behzad Djafari-Rouhani, Mohammad Farid, and Kaleem Raza Kazmi**, "Common solution to generalized mixed equilibrium problem and fixed point problem for a non-expansive semigroup in Hilbert space"
2:45 p.m.	**Miguel Argaez**, "Reduced-Order Modeling based on Wavelets for Large-Scale Problems"
3:15 p.m.	**Horacio Florez**, "Domain Decomposition Methods in Geomechanics"
3:45 p.m.	**Break**
4:15 p.m.	**Christian Servin and Vladik Kreinovich**, "Comparisons of Measurement Results as Constraints on Accuracies of Measuring Instruments: When Can We Determine the Accuracies from These Constraints?"
4:45 p.m.	**Vyacheslav V. Kalashnikov, Vladimir A. Bulavsky, and Nataliya I. Kalashnykova**, "Consistent Conjectures Are Optimal Nash Strategies in the Meta-Game"
5:15 p.m.	**Closing remarks**

A note on the fuzzy extension principle for LP problems with fuzzy coefficient matrix

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1 Introduction

Linear Programming (LP) is among the most used optimization models due to its efficiency, simplicity and reliability. In the last decades, fuzzy sets have been incorporated into LP models to represent imprecision coming from human being perceptions with successful results. Different approaches to Fuzzy Linear Programming (FLP) problems have been presented in bibliography, Rommelfanger [1–4], Ramík [5, 6], Ramík & Řimánek [7], Gasimov & Yenilmez [8], and Zimmermann [9, 10] who treated the field of classical fuzzy sets and its application in LP problems.

LP problems are composed by three sets of parameters: costs c, coefficient matrix A, and constraints b which are considered as deterministic. In FLP models, different combinations of fuzzy parameters namely $\tilde{c}, \tilde{A}, \tilde{b}$ can be considered into the analysis that lead to different optimization methods and routines. In this paper, we focus on the problem of having a fuzzy coefficient matrix $\tilde{A}$ and how the fuzzy extension principle operates into its optimal solution.

This paper focuses on how to compute the membership degree of a random realization of a_{ij} given previous experts perceptions and/or opinions about a_{ij} which are represented using fuzzy sets, and applied in LP problems. The paper is divided into six sections; a first one that introduces the main problem; a second section shows basics on fuzzy sets and matrices; a third one that presents the FLP with coefficient matrix model; a fourth section introduces the extension principle for fuzzy sets; a fifth section presents and application example, and finally some concluding remarks are shown in section 6.

2 Motivation

In practical applications, parameters $a_{ij} \in A$ are just realizations of a random process that can be represented through experts perceptions, so the analyst has to find a solution given those operation conditions. This way, there is a need for correlating practical issues (random realizations of a_{ij}) to experts perceptions and opinions about a_{ij}. This way, our problem regards to a mathematical programming model in the form $\text{Max}_x\{z = c'x : \tilde{A}x \leqslant b, x \geqslant 0\}$ where $\tilde{A}$ is a fuzzy coefficient matrix, and its fuzzy behavior.

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There is a family of problems that includes fuzzy coefficient matrices $\tilde{A}$; Fuzzy payoffs has been included in LP games problems by Bector & Chandra [11], and Campos [12], production planning with fuzzy coefficients has been treated by Pandiant [13], Gen et al. [14], Peidro et al. [15], and Figueroa-García et al. [16,17], fuzzy regression models have been proposed by Tanaka [18], Bargiela [19], etc. Most of their main results are based on measures that allow to find a single solution of the problem, but other possible choices are not considered into their analysis.

In practical scenarios theoretical solutions are unlikely to be applied since different configurations of parameters compose the problem, and its solution can differ from theoretical expectations which leads us to establish a way for analyzing the relationship between a random configuration and the theoretical solution of the problem.

On the other hand, there is a need for computing the membership degree that a random selection of coefficients of an LP problem has, to later compare it to what it is expected by experts. To do so, we analyze the problem from two points of view: a *constructive* approach that uses α-cuts to map imprecision around $\tilde{A}$, and a *deductive* approach that uses extension principle to see the behavior of a random selection of A_x enclosed into $\tilde{A}$, this is $A_x \in \textit{supp}(\tilde{A})$.

The deductive approach is based on the idea that a random selection/realization of A_x, $A_x \in \textit{supp}(\tilde{A})$ can occur anytime in practical scenarios e g. a realization of the payoffs of a game in a specific day, a particular consumption time, resources consumption and other production planning aspects for a specific period, random values of a regression model, etc. The main idea is to compute z_x^* which is the optimal solution of the LP for A_x, $z_x^* = \sup\{z_x = c'x : \ A_x x \leqslant b\}$ and project it over the fuzzy set of optimal solutions $\tilde{z}$ obtained from the constructive approach, using the extension principle as follows:

$$f(A_x)(z_x^*) \longrightarrow \tilde{z} \tag{1}$$

Then we analyze the relationship between both approaches through a lemma. Two application examples are presented, and the utility of the proposed lemma is shown. A comprehensive graph of our proposal is provided in Figure 1

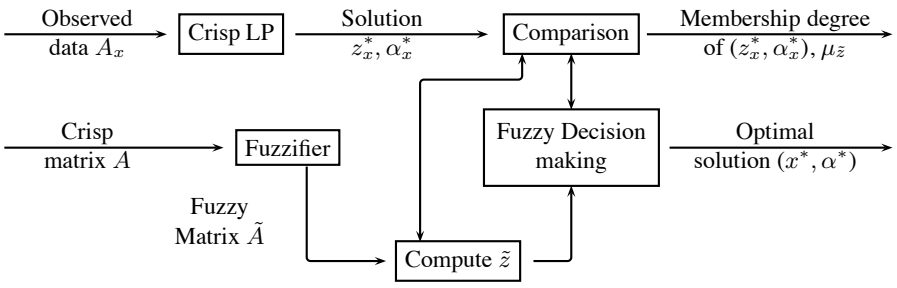

Fig. 1. Comparing observed values of A_x

Finally some concluding remarks of the obtained results are presented and some future work is proposed.

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Constraint Approach to Multi-Objective Optimization

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Abstract. In many practical situations, we would like to maximize (or minimize) several different criteria, and it is not clear how much weight to assign to each of these criteria. Such situations are ubiquitous and thus, it is important to be able to solve the corresponding multi-objective optimization problems. There exist many heuristic methods for solving such problems. In this paper, we reformulate multi-objective optimization as a constraint satisfaction problem, and we show that this reformulation explains two widely use multi-objective optimization techniques: optimizing a weighted sum of the objective functions and optimizing the product of normalized values of these functions.

1 Formulation of the Problem

Multi-objective optimization: examples. In many practical situations, we would like to maximize several different criteria.

For example, in meteorology and *environmental research*, it is important to measure fluxes of heat, water, carbon dioxide, methane and other trace gases that are exchanged within the atmospheric boundary layer. To perform these measurements, researchers build up vertical towers equipped with sensors at different heights; these tower are called *Eddy flux* towers. When selecting a location for the Eddy flux tower, we have several criteria to satisfy; see, e.g., [1, 5]: The station should located as far away as possible from roads, so that the gas flux generated by the cars do not influence our measurements of atmospheric fluxes. On the other hand, the station should be located as close to the road as possible, so as to minimize the cost of carrying the heavy parts when building such a station. The inclination at the station location should be small, because otherwise, the flux will be mostly determined by this inclination and will not be reflective of the atmospheric processes, etc.

In *geophysics*, different type of data provide complementary information about the Earth structures. For example, information from the body waves (P-wave receiver functions) mostly covers deep areas, while the information about the Earth surface is mostly contained in surface waves. To get a good understanding of the Earth structure, it is therefore important to take into account data of different types; see, e.g., [3, 9].

If we had only one type of data, then we can use the usual Least Squares approach $f_i(x) \to \min$ to find a model that best fits the data. If we knew

the relative accuracy of different data types, we could apply the Least Squares approach to all the data. In practice, however, we do not have a good information about the relative accuracy of different data types. In this situation, all we can say that we want to minimize the errors $f_i(x)$ corresponding to all the observations i.

Multi-objective optimization is difficult. The difficulty with this problem is that, in contrast to a simple optimization, the problem of multi-objective optimization is not precisely defined. Indeed, if we want to minimize a single objective $f(x) \to \min$, this has a very precise meaning: we want to find an alternative x_0 for which $f(x_0) \le f(x)$ for all other alternatives x. Similarly, if we want to maximize a single objective $f(x) \to \max$, this has a very precise meaning: we want to find an alternative x_0 for which $f(x_0) \ge f(x)$ for all other alternatives x.

In contrast, for a multi-objective optimization problem

$$f_1(x) \to \min; \quad f_2(x) \to \min; \quad \ldots; \quad f_n(x) \to \min \tag{1}$$

or

$$f_1(x) \to \max; \quad f_2(x) \to \max; \quad \ldots; \quad f_n(x) \to \max, \tag{2}$$

no such precise meaning is known.

Let us illustrate this ambiguity on the above trip example. In many cases, convenient direct flights which save on travel time are more expensive, while a cheaper trip may involve a long stay-over in between flights. So, if we find a trip that minimizes cost, the trip takes longer. Vice versa, if we minimize the travel time, the trip costs more.

It is therefore necessary to come up with a way to find an appropriate compromise between several objectives.

2 Analysis of the Problem and Two Main Ideas

Analysis of the problem. Without losing generality, let us consider a multi-objective maximization problem. In this problem, ideally, we would like to find an alternative x_0 that satisfies the constraints $f_i(x_0) \ge f_i(x)$ for all objectives i and for all alternatives x. In other words, in the ideal case, if we select an alternative x at random, then with probability 1, we satisfy the above constraint.

Main ideas. The problem is that we cannot satisfy all these constraints with probability 1. A natural idea is thus to find x_0 for which the probability of satisfying these constraints is as high as possible. Let us describe two approaches to formulating this idea (i.e., the corresponding probability) is precise terms.

First approach: probability to satisfy all n constraints. The first approach is to look for the probability that for a randomly selected alternative x, we have $f_i(x_0) \ge f_i(x)$ *for all* i.

Second approach: probability to satisfy a randomly selected constraint. An alternative approach is to look for the probability that for a randomly selected alternative x and *for a randomly selected objective* i, we have $f_i(x_0) \geq f_i(x)$.

How to formulate these two ideas in precise terms. To formulate the above two ideas in precise terms, we need to estimate two probabilities:

- the probability $p_I(x_0)$ that for a randomly selected x, we have $f_i(x_0) \geq f_i(x)$ for all i, and
- the probability $p_{II}(x_0)$ that for a randomly selected x and a randomly selected i, we have $f_i(x_0) \geq f_i(x)$.

Let us estimate the first probability. Since we do not have any prior information about the dependence between different objective functions $f_i(x)$ and $f_j(x)$, $i \neq j$, it is reasonable to assume that the events $f_i(x_0) \geq f_i(x)$ and $f_j(x_0) \geq f_j(x)$ are independent for different i and j. Thus, the desired probability $p_I(x_0)$ that all n such inequalities are satisfied can be estimated as the product $p_I(x_0) = \prod_{i=1}^{n} p_i(x_0)$ of n probabilities p_i of satisfying the corresponding inequalities.

So, to estimate p, it is sufficient to estimate, for every i, the probability $p_i(x_0)$ that $f_i(x_0) \geq f_i(x)$ for a randomly selected alternative x.

How can we estimate this probability $p_i(x_0)$? Again, in general, we do not have much prior knowledge of the i-th objective function $f_i(x)$. What do we know? Before starting to solve this problem as a multi-objective optimization problem, we probably tried to simply optimize each of the objective functions – hoping that the corresponding solution would also optimize all other objective functions. Since we are interesting in maximizing, this means that we know the largest possible value M_i of each of the objective functions: $M_i = \max_x f_i(x)$.

In many practical cases, the optimum can be attained by differentiating the objective function and equating all its derivatives to 0. This is, for example, how the Least Squares method works: to optimize the quadratic function that describes how well the model fits the data, we solve the system of linear equations obtained by equating all partial derivatives to 0. It is important to mention that when we consider the points where all the partial derivatives are equal to 0, we find not only maxima but also minima of the objective function. Thus, it is reasonable to assume that in the process of maximizing each objective function $f_i(x)$, in addition to this function's maximum, we also compute its minimum $m_i = \min_x f_i(x)$.

Since we know the smallest possible value m_i of the objective function $f_i(x)$, and we know its largest possible value M_i, we thus know that the value $f_i(x)$ corresponding to a randomly selected alternative x must lie inside the interval $[m_i, M_i]$.

In effect, this is all the information that we have: that the random value $f_i(x)$ is somewhere in the interval $[m_i, M_i]$. Since we do not have any reason to

believe that some values from this interval are more probable and some values are less probable, it is reasonable to assume that all the values from this interval are equally probable, i.e., that we have a uniform distribution on the interval $[m_i, M_i]$.

This argument – known as Laplace Indeterminacy Principle – can be formalized as selecting the distribution with the probability density $\rho(x)$ for which the entropy $S = -\int \rho(x) \cdot \ln(\rho(x))\, dx$ is the largest possible. One can check that for distributions on the given interval, the uniform distribution is the one with the largest entropy [6].

For the uniform distribution on the values $f_i(x) \in [m_i, M_i]$, the probability $p_i(x_0)$ that the random value $f_i(x)$ does not exceed $f_i(x_0)$, i.e., belongs to the subinterval $[m_i, f_i(x_0)]$, is equal to the ratio of the corresponding intervals, i.e., to $p_i(x_0) = \dfrac{f_i(x_0) - m_i}{M_i - m_i}$. Thus, the desired probability $p_I(x_0)$ is equal to the product of such probabilities. So, we arrive at the following precise formulation of the first idea:

Precise formulation of the first idea. To solve a multi-objective optimization problem (2), we find a value x_0 for which the product $p_I(x_0) = \prod\limits_{i=1}^{n} \dfrac{f_i(x_0) - m_i}{M_i - m_i}$ attains the largest possible value, where $m_i \stackrel{\text{def}}{=} \min\limits_x f_i(x)$ and $M_i \stackrel{\text{def}}{=} \max\limits_x f_i(x)$.

Let us estimate the second probability. In the second approach, we select the objective function f_i at random. Since we have no reason to prefer one of the n objective functions, it makes sense to select each of these n functions with equal probability $\dfrac{1}{n}$.

For each selection of the objective function i, we know the probability $p_i(x_0) = \dfrac{f_i(x_0) - m_i}{M_i - m_i}$ that we will have $f_i(x_0) \geq f_i(x)$ for a randomly selected alternative x. The probability of selecting each objective function $f_i(x)$ is equal to $\dfrac{1}{n}$. Thus, we can use the complete probability formula to compute the desired probability $p_{II}(x_0)$:

Precise formulation of the second idea. To solve a multi-objective optimization problem (2), we find a value x_0 for which the expression $p_{II}(x_0) = \sum\limits_{i=1}^{n} \dfrac{1}{n} \cdot \dfrac{f_i(x_0) - m_i}{M_i - m_i}$ attains the largest possible value.

Discussion. Let us show that both ideas lead to known (and widely used) methods for solving multi-objective optimization problems.

The second idea leads to optimizing a linear combination of objective functions. Let us start with analyzing the second idea, since the resulting formula with the sum looks somewhat simpler than the product-based formula corresponding to the first idea.

In the case of the second idea, the optimized value $p_{II}(x_0)$ is a linear combination of n objective functions – to be more precise, it is an arithmetic average

of the objective functions normalized in such a way that their values are within the interval $[0,1]$: $p_{II}(x_0) = \dfrac{1}{n} \cdot \sum_{i=1}^{n} f_i'(x_0)$, where $f_i'(x_0) \stackrel{\text{def}}{=} \dfrac{f_i(x_0) - m_i}{M_i - m_i}$.

Maximizing a linear combination of the objective functions is indeed the most widely used approach to solving multi-objective optimization problems; see, e.g., [4].

The first idea leads to maximizing a product of (normalized) objective functions. One can easily see that the first idea leads to maximizing a product of normalized objective functions: $p_I(x_0) = \prod_{i=1}^{n} f_i'(x_0)$.

Maximizing such a product is exactly what Bellman-Zadeh fuzzy approach recommends (if we use the product as an "and" operation); see, e.g., [2, 8]. It fits will with our own proposal for such a situation; see, e.g., [5].

This is also exactly what the the Nobelist John Nash recommended for a similar situation of making a group decision when each participant would like to optimize his/her own utility $f_i(x) \to \max$; see, e.g., [7].

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Using Interval Constraint Solving Techniques in Dynamic Systems Behavior Prediction

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Measurements from observation of a natural phenomenon should help us identify important features of it. For instance, characteristics such as the initial size of a population within the observation of any species, the rate of its reproduction [5], the interaction with any competitive species [6], or what the maximum number of individuals it can support is, shoud be easily identified. Indeed, such features / characteristics can be obtained as the solution of a, sometimes nonlinear, system of equations, which could arise from the discretization of a set of partial differential equations [2] in which measurements have been entered as solutions as particular instants of time.

However, in real life, we do not have exact values as a result of measurements / observations, because measurements always involve some level of uncertainty. As a result, we need to use another strategy than traditional PDE or nonlinear systems solving, to handle the uncertainty. One way of handling uncertainty [3] is to use intervals and a way of handling intervals in the solution of nonlinear systems is via constraint solving techniques [1, 4].

In this work, we show how intervals and constraint techniques can be used to identify dynamic phenomena characteristics as well as to predict their future behavior, and what we obtain from doing so. We report and analyze the result of experiments carried out on different PDE problems. In particular, we study the efficiency of our approach subject to the number of measurements and their distance in time.

We discuss the current limitations and challenges of this approach, and draw directions for future work.

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Global Optimization via Speculation

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Abstract. Most electronic devices we employ in our every day lives, such as cell phones and computers, require careful design and assembly. The micro-components inside them must be arranged and connected in a very small space. For the designer, this is a challenging process, as the machines that create and assemble these components limit the size and distance between components. Many devices employ components that operate at a certain speed. The design of portable devices, such as mobile phones, also require that the device consumes the minimum amount of electricity possible, so the device can keep functioning for a longer time without recharging its battery. The engineers creating these devices could try multiple designs and see which one is best. However, this unstructured approach would be inefficient, time-consuming, and likely not to guarantee that the best configuration has been identified.
Instead, we can express the properties of the design and its restrictions as a series of mathematical equations, with parameters, such as size and location, and the energy consumed as an objective function to be minimized. This is an optimization problem, a category of problems in which we seek the optimum (maximum or minimum) value of an objective function, while possibly meeting some constraints / requirements.
We solve optimization problems using search algorithms. Local search focuses on making slight adjustments to the variables continuously improving the objective function until these parametric adjustments do not yield a better objective value. Local search algorithms rely on an initial guess of the parameter values to converge on the closest optimum. Because of the limited range of the search in these algorithms, they cannot guarantee that the solution they find is the best of all values (a global optimum). Global search requires techniques that expand the range of the search to include the entire domain to overcome this drawback. Such techniques trade off execution time for increased accuracy. Global algorithms take longer than local search, but can guarantee that their solution is the global minimum.
In this work, we recall state of the art techniques for global optimization and present a new global search algorithm. Using intervals and interval constraint solving techniques, our algorithm conducts an exhaustive search, guaranteeing the global optimality of the solution(s). The algorithm uses speculation over the interval range of the objective function, by placing "bets" on the expected minimum value until finding it or proving it is not correct. In particular, we present the general structure of this speculative algorithm and the techniques used to improve its performance while maintaining a guarantee on global optimality. Numerical results are presented and discussed.

Keywords: Optimization, Interval Computations, Optimization under Constraints

Optimizing pred(25) Is NP-Hard

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Abstract. Usually, in data processing, to find the parameters of the models that best fits the data, people use the Least Squares method. One of the advantages of this method is that for linear models, it leads to an easy-to-solve system of linear equations. A limitation of this method is that even a single outlier can ruin the corresponding estimates; thus, more robust methods are needed. In particular, in software engineering, often, a more robust pred(25) method is used, in which we maximize the number of cases in which the model's prediction is within the 25% range of the observations. In this paper, we show that even for linear models, pred(25) parameter estimation is NP-hard.

1 Formulation of the Problem

Need to estimate parameters of models. In many practical situations, we know that a quantity y depends on the quantities $x_1, \ldots, x_n$, and we know the general type of this dependence. In precise terms, this means that we know a family of functions $f(c_1, \ldots, c_p, x_1, \ldots, x_n)$ characterized by parameters c_i, and we know that the actual dependence corresponds to one of these functions.

For example, we may know that the dependence is linear; in this cases, the corresponding family takes the form

$$f(c_1, \ldots, c_n, c_{n+1}, x_1, \ldots, x_n) = c_{n+1} + \sum_{i=1}^{n} c_i \cdot x_i.$$

In general, we know the type of the dependence, but we do not know the actual values of the parameters. These values can only be determined from the measurements and observations, when we observe the values x_j and the corresponding value y. Measurement and observations are always approximate, so we end up with tuples $(x_{1k}, \ldots, x_{nk}, y_k)$, $1 \le k \le K$, for which $y_k \approx f(c_1, \ldots, c_p, x_{1k}, \ldots, x_{nk})$ for all k. We need to estimate the parameters $c_1, \ldots, c_p$ based on these measurement results.

Least Squares: traditional way of estimating parameters of models. In most practical situations, the Least Squares method is used to estimate the desired parameters. In this method, we select the values c_i for which the sum of the squares of the approximation errors is the smallest possible:

$$\sum_k (y_k - f(c_1, \ldots, c_p, x_{1k}, \ldots, x_{nk}))^2 \to \min_{c_1, \ldots, c_p}.$$

One of advantages of this approach is that, when the model $f(c_1, \ldots, c_p, x_1, \ldots, x_n)$ linearly depends on the parameters c_i, the sum of squares is a quadratic function of c_i. Thus, when we apply the usual criterion for the minimum – differentiate the sum with respect to each variable x_i and equate all the resulting partial derivatives to 0 – we get a system of linear equations, from which we can easily find all the unknown $c_1, \ldots, c_p$.

Least Squares is not always the optimal way of estimating the parameters. The Least Squares approach known to be optimal for the case when all the approximation errors $y_k - f(c_1, \ldots, c_p, x_{1k}, \ldots, x_{nk})$ are independent and all distributed according to the same normal distribution. In practice, however, we often have outliers – e.g., values corresponding to the malfunction of a measuring instrument – and in the presence of even a single outlier, the Least Squares method can give very wrong results.

Let us illustrate this on the simplified example, when y does not depend on any variables x_i at all, i.e., when $y = c$ for some unknown constant c. In this case, we need to estimate the value c based on the observations $y_1, \ldots, y_K$. For this problem, the Least Squares method takes the form $\sum\limits_{k=1}^{K} (y_k - c)^2 \to \min$. Differentiating the sum with respect to the unknown c and equating the derivative to 0, we conclude that $c = \dfrac{y_1 + \ldots + y_K}{K}$.

This formula works well if all the values y_i are approximately equal to c. For example, if the actual value of c is 0, and $|y_i| \le 0.1$, we get an estimate $|c| \le 0.1$. However, if out of 100 measurements y_i, one of an outlier equal to 1000, the estimate becomes close to 10 – and thus, far away from the actual value 0.

To take care of such situations, we need estimates which do not change as much in the presence of possible outliers. Such methods are called *robust* [2].

pred(25) as an example of a robust estimate. One of the possible robust estimates consists of selecting a percentage α and selecting the values of the parameters for which the number of observations for which the prediction is within α% from the observed value is the largest possible. In other words, each prediction is formulated as a constraint, and we look for parameters that maximize the number of satisfied constraint. This technique is known as pred(α).

This method is especially widely used in software engineering, e.g., for estimating how well different models can predict the overall software effort and/or the number of bugs. In software engineering, this method is most frequently applied as pred(25), for $\alpha = 25$; see, e.g., [1, 3].

Problem. In contrast to the Least Squares approach, for which the usual calculus ideas lead to an efficient optimization algorithm, no such easy solution is known for pred(25) estimates; all known algorithms for this estimation are rather time-consuming. A natural question arises: is this because we have not yet found a feasible algorithm for computing these estimates, or is this estimation problem really hard?

What we prove in this paper. In this paper, we prove that even for a linear model with no free term c_{n+1}, pred(25) estimation – as well as pred(α) estimation

for any $\alpha > 0$ – is an NP-hard problem. In plain terms, this means that this problem is indeed inherently hard.

2 Main Result and Its Proof

Definition 1. *Let $\alpha \in (0,1)$ be a rational number. By a* linear pred(α)-estimation problem*, we means the following problem:*

- Given: *an integer n, K rational-valued tuples $(x_{1k}, \ldots, x_{nk}, y_k)$, $1 \le k \le K$, and an integer $M < K$;*
- Check: *whether there exist parameters $c_1, \ldots, c_n$ for which in at least M cases k, we have*

$$\left| y_k - \sum_{i=1}^{n} c_i \cdot x_{ik} \right| \le \alpha \cdot \left| \sum_{i=1}^{n} c_i \cdot x_{ik} \right|.$$

Proposition 1. *For every α, the linear pred(α)-estimation problem is NP-hard.*

Proof. To prove this result, we will reduce, to this problem, a known NP-hard problem of checking whether a set of integer weights $s_1, \ldots, s_m$ can be divided into two parts of equal overall weight, i.e., whether there exist integers $y_j \in \{-1, 1\}$ for which $\sum\limits_{j=1}^{m} y_j \cdot s_j = 0$; see, e.g., [4].

In the reduced problem, we will have $n = m + 1$, with $n = m + 1$ unknown coefficients $c_1, \ldots, c_m, c_{m+1}$. The parameters c_i will correspond to the values y_i, and c_{m+1} is equal to 1. We will build tuples corresponding to equations $y_i = 1$ and $y_i = -1$ for $i \le m$, to $c_{m+1} = 1$, and to the equation $c_{m+1} + \sum\limits_{i=1}^{m} y_i \cdot s_i = 1$.

To each equation of the type $y_i = 1$ or $c_{m+1} = 1$, we put into correspondence the following two tuples:

- In the first tuple, $x_{ik} = 1 + \varepsilon$, $x_{jk} = 0$ for all $j \ne i$, and $y_k = 1$. The resulting linear term has the form $c_i \cdot (1 + \varepsilon)$ and thus, the corresponding inequality takes the form $1 - \varepsilon \le (1 + \varepsilon) \cdot c_i \le 1 + \varepsilon$, i.e., equivalently, the form $\dfrac{1 - \varepsilon}{1 + \varepsilon} \le c_i \le 1$.
- In the second tuple, $x_{ik} = 1 - \varepsilon$, $x_{jk} = 0$ for all $j \ne i$, and $y_k = 1$. The resulting linear term has the form $c_i \cdot (1 - \varepsilon)$ and thus, the corresponding inequality takes the form $1 - \varepsilon \le (1 - \varepsilon) \cdot c_i \le 1 + \varepsilon$, i.e., equivalently, the form $1 \le c_i \le \dfrac{1 + \varepsilon}{1 - \varepsilon}$.

It should be mentioned that the only value c_i that satisfies both inequalities is the value $c_i = 1$.

Similarly, to each equation of the type $y_i = -1$, we put into correspondence following two tuples.

– In the first tuple, $x_{ik} = 1 + \varepsilon$, $x_{jk} = 0$ for all $j \neq i$, and $y_k = -1$. The resulting linear term has the form $c_i \cdot (1 + \varepsilon)$ and thus, the corresponding inequality takes the form $-1 - \varepsilon \leq (1 + \varepsilon) \cdot c_i \leq -1 - \varepsilon$, i.e., equivalently, the form $-1 \leq c_i \leq -\dfrac{1 - \varepsilon}{1 + \varepsilon}$.
– In the second tuple, $x_{ik} = 1 - \varepsilon$, $x_{jk} = 0$ for all $j \neq i$, and $y_k = -1$. The resulting linear term has the form $c_i \cdot (1 - \varepsilon)$ and thus, the corresponding inequality takes the form $-1 - \varepsilon \leq (1 - \varepsilon) \cdot c_i \leq -1 + \varepsilon$, i.e., equivalently, the form $-\dfrac{1 + \varepsilon}{1 - \varepsilon} \leq c_i \leq -1$.

Here also, the only value c_i that satisfies both inequalities is the value $c_i = -1$.

Finally, to the equation $c_{m+1} + \sum\limits_{j=1}^{m} y_j \cdot s_j = 1$, we put into correspondence the following two tuples. In both tuples, $y_k = 1$.

– In the first tuple, $x_{ik} = (1 + \varepsilon) \cdot s_i$, and $x_{m+1,k} = 1 + \varepsilon$. The corresponding linear term has the form $(1 + \varepsilon) \cdot \left(\sum\limits_{i=1}^{m} c_i \cdot s_i + c_{m+1} \right)$, and thus, the corresponding inequality takes the form

$$1 - \varepsilon \leq (1 + \varepsilon) \cdot \left(\sum_{i=1}^{m} c_i \cdot s_i + c_{m+1} \right) \leq 1 + \varepsilon,$$

i.e., equivalently,

$$\frac{1 - \varepsilon}{1 + \varepsilon} \leq \sum_{i=1}^{m} c_i \cdot s_i + c_{m+1} \leq 1.$$

– In the second tuple, $x_{ik} = (1 - \varepsilon) \cdot s_i$, and $x_{m+1,k} = 1 - \varepsilon$. The corresponding linear term has the form $(1 - \varepsilon) \cdot \left(\sum\limits_{i=1}^{m} c_i \cdot s_i + c_{m+1} \right)$, and thus, the corresponding inequality takes the form

$$1 - \varepsilon \leq (1 - \varepsilon) \cdot \left(\sum_{i=1}^{m} c_i \cdot s_i + c_{m+1} \right) \leq 1 + \varepsilon,$$

i.e., equivalently,

$$1 \leq \sum_{i=1}^{m} c_i \cdot s_i + c_{m+1} \leq \frac{1 + \varepsilon}{1 - \varepsilon}.$$

Here, both inequalities are satisfied if and only if $\sum\limits_{i=1}^{m} c_i \cdot s_i + c_{m+1} = 1$.

Overall, we have $2m + 2$ pairs, i.e., $4m + 4$ tuples. If for the given values $s_1, \ldots, s_m$, the original NP-hard problem has a solution y_i, then we can take $c_i = y_i$, $c_{m+1} = 1$, and thus satisfy $M \stackrel{\text{def}}{=} 2m + 4$ inequalities. Let us show that, vice versa, if at least $2m+4$ inequalities are satisfied, this means that the original problem has a solution.

Indeed, for every i, each of the two inequalities corresponding to $y_i = 1$ implies that $c_i > 0$ while each of the two inequalities corresponding to $y_i = -1$ implies that $c_i < 0$. Thus, these inequalities incompatible, which means that for every i, at most two inequalities can be satisfied. If for some i, fewer than two inequalities are satisfied, then even when for every $j \neq i$, we have two, and all four remaining inequalities are satisfied, we will still have fewer than $2m + 4$ satisfied inequalities. This means that if $2m+4$ inequalities are satisfied, then for every i, two inequalities are satisfied – and thus, either $c_i = 1$ or $c_i = -1$. Now, the four additional inequalities also have to be satisfied, so we have $c_{m+1} = 1$, and $\sum_{i=1}^{m} c_i \cdot s_i + c_{m+1} = 1$, hence $\sum_{i=1}^{m} c_i \cdot s_i = 0$. The reduction is proven, and thus our problem is indeed NP-hard.

Comment. In this proof, we consider situations in which about half of the inequalities are satisfied. We may want to restrict ourselves to situations in which a certain proportion of inequality should be satisfied – e.g., 90% or 99%. With such a restriction, the problem remains NP-hard.

To prove this, it is sufficient to consider a similar reduction, in which:

- instead of single pair of tuples corresponding to $c_{m+1} = 1$ we have N identical pairs (for a sufficiently large N), and similarly,
- instead of a single pair corresponding to the equation $\sum_{j=1}^{m} y_j \cdot s_j = 0$, we have N such identical pairs.

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Common solution to generalized mixed equilibrium problem and fixed point problem for a nonexpansive semigroup in Hilbert space

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Abstract

Throughout the paper unless otherwise stated, H denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H.

Recall that a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C$.

A family $\Im := \{T(s) : 0 \leq s < \infty\}$ of mappings from C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

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(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

The common fixed point set of $\Im$ is denoted by $\mathrm{Fix}(\Im)$, i.e.,

$$\mathrm{Fix}(\Im) := \{x \in C : T(s)x = x, 0 \leq s < \infty\} = \bigcap_{0 \leq s < \infty} \mathrm{Fix}(T(s)),$$

where $\mathrm{Fix}(T(s))$ is the set of fixed points of $T(s)$.

Recall that a mapping $f : C \to C$ is said to be weakly contractive [16] if

$$\|f(x) - f(y)\| \leq \|x - y\| - \psi(\|x - y\|), \ \forall x, y \in C,$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and strictly increasing function such that ψ is positive on $(0, +\infty)$ and $\psi(0) = 0$. If $\psi(t) = (1-k)t$ with $0 < k < 1$, then f is said to be contractive with constant $k > 0$. If $\psi(t) = 0$, then f is said to be nonexpansive.

The fixed point problem (in short, FPP) for a nonexpansive semigroup S is: Find $x \in C$ such that

$$x \in \mathrm{Fix}(S). \tag{1.1}$$

Next, we consider the following generalized mixed equilibrium problem (in short, GMEP): Find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x\rangle + \phi(y, x) - \phi(x, x) \geq 0, \quad \forall y \in C, \tag{1.2}$$

where $\mathbb{R}$ is the set of all real numbers, and $F : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ are nonlinear bifunctions, and $A : H \to H$ is a suitable nonlinear operator. The solution set of GMEP(1.2) is denoted by Sol(GMEP(1.2)).

An operator $B : H \to H$ is said to be strongly positive if there exists a constant $\bar{\gamma} > 0$ such that $< Bx, x > \geq \bar{\gamma}\|x\|^2, \forall x \in H$.

In 2006, Marino and Xu [12] introduced the following implicit and explicit iterative methods based on viscosity approximation method, for the fixed point problem (FPP) for a nonexpansive self mapping T of H:

$$x_t = t\gamma f(x_t) + (I - tB)Tx_t, \tag{1.6}$$

and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)Tx_n, \tag{1.7}$$

where f is a contraction mapping on H with constant $\alpha > 0$, B is a strongly positive self-adjoint and bounded linear operator on H with constant $\bar{\gamma} > 0$ and $\gamma \in (0, \frac{\bar{\gamma}}{\alpha})$. They proved that the net (x_t) and the sequence $\{x_n\}$ generated by (1.6) and (1.7), respectively, both converge strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in \text{Fix}(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \text{Fix}(T)} \frac{1}{2}\langle Bx, x \rangle - h(x),$$

where h is the potential function for γf.

Recently, Ceng *et al.* [3] introduced and studied the following explicit iterative scheme for FPP for a nonexpansive mapping T:

$$x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)Tx_n], \tag{1.8}$$

where P_C is the metric projection on C and $\mu > 0$.

In 2008, Plubtieng and Punpaeng [14] introduced and studied the following implicit iterative scheme to prove a strong convergence theorem for FPP(1.1):

$$x_t = tf(x_t) + (1 - t)\frac{1}{s_t}\int_0^{s_t} T(s)x_t ds, \tag{1.9}$$

where (x_t) is a continuous net and (s_t) is a positive real net tending to infinity.

Very recently, Xiao *et al.* [16] introduced and studied the following implicit iterative scheme and obtained a strong convergence theorem for EP(1.4) and FPP(1.1):

$$\begin{cases} F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall\ y \in C, \\ z_n = (I - \beta_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)u_n ds + \beta_n u_n, \\ x_n = (1 - \alpha_n A)z_n + \alpha_n \gamma f(x_n), \quad \forall\ n \geq 1. \end{cases} \tag{1.12}$$

Motivated by the work of Ceng *et al.* [3], Xiao *et al.* [16], Cianciaruso *et al.* [5], Kazmi *et al.* [9, 10, 11], Djafari Rouhani *et al.* [6], and by the ongoing research in this direction, we suggest and analyze an explicit hybrid relaxed extragradient iterative method

for approximating the common solution to the generalized mixed equilibrium problem and fixed point problem for a nonexpansive semigroup in Hilbert space. Further, we prove that the sequence generated by the proposed iterative scheme converges strongly to the common solution of the generalized mixed equilibrium problem and fixed point problem for the nonexpansive semigroup. This common solution is the unique solution of a variational inequality problem and is the optimality condition for a minimization problem. The results and methods presented here improve and generalize the corresponding results and methods given in [5, 10, 6, 11, 16].

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ReducedOrder Modeling based on Wavelets for Large-Scale Problems

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Domain Decomposition Methods in Geomechanics

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Abstract. Hydrocarbon production or injection of fluids in the reservoir can produce changes in the rock stresses and in-situ geomechanics, potentially leading to compaction and subsidence with harmful effects in wells, cap-rock, faults, and the surrounding environment as well. In order to tackle these changes and their impact, accurate simulations are essential.

The Mortar Finite Element Method (MFEM) is a powerful technique in order to formulate a weak continuity condition at the interface of subdomains in which different meshes, i.e. non-conforming or hybrid, and/or variational approximations are used. This is particularly suitable when coupling different physics on different domains, such as elasticity and poroelasticity, in the context of coupled flow and geomechanics.

We implement popular Domain Decomposition (DD) techniques in order to carry large simulations, by taking full advantage of current parallel computer architectures. Different solution schemes can be defined depending upon the way information is exchanged between subdomain interfaces. Three different schemes, i.e. Dirichlet-Neumann (D-N), Neumann-Neumann (N-N) and MFEM, are tested and the advantages and disadvantages of each of them are identified. The MFEM is extended to deal with curve interfaces represented by Non-Uniform Rational B-Splines (NURBS) curves and surfaces. The goal is to have a more robust geometrical representation for mortar spaces, which allows gluing non-conforming interfaces on realistic geometries. The resulting mortar saddle-point problem is decoupled by means of the D-N and N-N DD-schemes.

A reservoir geometry reconstruction procedure based on NURBS surfaces is presented as well. The technique builds a robust piecewise continuous geometrical representation that can be exploited by MFEM in order to tackle realistic problems. Tensor product meshes are usually propagated from the reservoir in a conforming way into its surroundings, which makes non-matching interfaces highly attractive in this case.

Several examples of coupling of elasticity and poroelasticity, ranging from near-wellbore applications to field level subsidence computation, show that the proposed approach can handle problems of practical interest. In order to facilitate the implementation of complex workflows, an advanced Python wrapper interface that allows programming capabilities is also implemented. The proposed serial-parallel approach seems to be appropriate to handle geomechanical problems involving different meshes for

flow and mechanics as well as coupling parallel mechanistic codes with legacy flow simulators.

1 Mathematical Model

This section discusses the governing equations for linear homogeneous isotropic poro-elasticity and their finite element formulation. We omit details for the sake of brevity, a more detailed treatment can be found in [33,10,34,35,1,6]. We consider a bounded domain $\Omega \subset \mathbb{R}^2$ and its boundary is $\Gamma = \partial\Omega$. Let ξ_h be a non-degenerate, quasi-uniform conforming partition of Ω composed of triangles or quadrilaterals. It can be shown that [2], for deformable porous media, the single-phase flow model derives from the continuity equation for incompressible flow and Darcy's law, yielding:

$$\frac{\partial \phi^*}{\partial t} + \nabla \cdot \left(-\frac{1}{\mu} \underline{\underline{K}} \left(\nabla p - \rho g \nabla z \right) \right) = q \tag{1}$$

where ϕ^* is model specific porosity, $\underline{\underline{K}}$ is the absolute permeability tensor, μ is the dynamic viscosity, ρ is the fluid density, g is the gravity acceleration constant, p is the fluid pressure and q represents sources and sinks. The porosity ϕ^* is given by:

$$\phi* = \phi^0 + \alpha \cdot \left(\nabla \cdot \underline{u} - \varepsilon_v^0 \right) + \frac{1}{M} \left(p - p^0 \right) \tag{2}$$

where α is the Biot's constant, $\underline{u}$ is the displacement vector, ε_v^0 is the initial volumetric strain, M is the Biot's modulus [36], ϕ^0 and p^0 account for a reference or initial state. The typical boundary conditions (BC) for pressure involve Neumann or no-flow namely:

$$\nabla p \cdot \underline{\hat{n}} = 0 \text{ on } \Gamma, \tag{3}$$

plus an initial pressure distribution in the domain. Sources and sinks allow the representation of injector and producer wells, respectively. Here $\underline{\hat{n}}$ is the outer unitary normal vector. For the elasticity part, we start from the equilibrium equation for a quasi-steady process, which means we discard the acceleration term:

$$\begin{aligned} -\nabla \cdot \underline{\underline{\sigma}} &= \underline{f} \text{ in } \Omega \; ; \; \Gamma = \Gamma_D^u \cup \Gamma_N^u \\ \underline{u} &= \underline{0} \text{ on } \Gamma_D^u \\ \underline{t} &= \underline{\underline{\sigma}} \cdot \underline{\hat{n}} \text{ on } \Gamma_N^u \end{aligned} \tag{4}$$

where $\underline{\underline{\sigma}}$ is the stress tensor, $\underline{f}$ is the vector of body forces, such as gravity for instance. BC usually involve prescribed tractions on part of the boundary. Boundary conditions can be decomposed in Dirichlet type BC, i.e. Γ_D^u, and Neumann type BC, i.e Γ_N^u, where the external tractions are prescribed. Hooke's law and Biot's poroelastic theory define $\underline{\underline{\sigma}}$ by:

$$\underline{\underline{\sigma}} = \underline{\underline{C}} : \underline{\underline{\varepsilon}} - \alpha \left(p - p^0 \right) \underline{\underline{\delta}} \; ; \; \underline{\underline{C}} = \lambda \underline{\underline{\delta}} \otimes \underline{\underline{\delta}} + 2\mu \underline{\underline{I}} \tag{5}$$

where $\underline{\underline{C}}$ is the elastic moduli for isotropic elasticity, $\underline{\underline{\delta}}$ is the Kroneker delta while λ, μ, are the Lamé constants, and $\underline{\underline{I}}$ is the fourth-order identity tensor. We derive a weak form by substituting Eq. (2) into Eq. (1) and multiplying by a test function $v \in H_0^1(\Omega)$ and integrating over the domain and applying the Gauss-divergence theorem:

$$\begin{aligned} &\int_{\Omega} \left(\tfrac{1}{M} \tfrac{\partial p}{\partial t} v + \alpha v \nabla \cdot \underline{\dot{u}} + \tfrac{1}{\mu} \underline{\underline{K}} \cdot \nabla p (\nabla v)^T \right) \cdot dx = \int_{\Omega} q \cdot v dx + \\ &\int_{\Omega} \left(\tfrac{\rho g}{\mu} \underline{\underline{K}} \cdot \nabla z (\nabla v)^T \right) dx + \int_{\partial \Omega_N^p} v \tfrac{1}{\mu} \underline{\underline{K}} (\nabla p - \rho g \nabla z) \cdot \hat{\underline{n}}^T ds \end{aligned} \tag{6}$$

A weak form for the equilibrium Eq. (4) can be derived in a similar way, by testing against a given virtual displacement. We arrive at:

$$\int_{\Omega} (\nabla \underline{\eta})^T : \underline{\underline{\sigma}} d\Omega = \int_{\partial \Omega_N^u} \underline{\eta}^T \cdot \underline{t} ds + \int_{\Omega} \underline{\eta}^T \cdot \underline{f} d\Omega \tag{7}$$

where $\underline{t} = \underline{\underline{\sigma}} \cdot \hat{\underline{n}}$ are the tractions applied as Neumman BC. This is the well-known virtual work statement. Finally substituting the generalized Hooke's law Eq. (5) into Eq. (7) and using Eq. (6) leads to our finite element model for linear isotropic poroelasticity, thus:

$$\begin{bmatrix} 0 & 0 \\ \underline{\underline{\mathrm{Q}}}^T & \underline{\underline{\mathrm{S}}} \end{bmatrix} \frac{d}{dt} \begin{Bmatrix} \underline{u} \\ \underline{p} \end{Bmatrix} + \begin{bmatrix} \underline{\underline{\mathrm{K}}} & -\underline{\underline{\mathrm{Q}}} \\ 0 & \underline{\underline{\mathrm{H}}} \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{p} \end{Bmatrix} = \begin{Bmatrix} \underline{f_u} \\ \underline{f_p} \end{Bmatrix} \tag{8}$$

where the matrices are given by:

$$\begin{aligned} &\underline{\underline{\mathrm{S}}} = \int_{\Omega} \tfrac{1}{M} \underline{\Pi} \cdot \underline{\Pi}^T dx \; ; \; \underline{\underline{\mathrm{Q}}} = \int_{\Omega} \underline{\underline{B}}^T \alpha \underline{m} \cdot \underline{\Pi} dx \\ &\underline{\underline{\mathrm{K}}} = \int_{\Omega} \underline{\underline{B}}^T \underline{\underline{C}} \, \underline{\underline{B}} dx \; ; \; \underline{f_u} = \int_{\partial \Omega_N^u} \underline{t} \cdot \underline{\underline{\Psi}}^T ds + \int_{\Omega} \underline{\underline{\Psi}}^T \underline{f} \cdot dx \\ &\underline{\underline{\mathrm{H}}} = \int_{\Omega} \tfrac{1}{\mu} \underline{\underline{K}} \nabla \underline{\Pi} \cdot (\nabla \underline{\Pi})^T dx \; ; \; \underline{m} = (1,1,0)^T \\ &\underline{f_p} = \int_{\partial \Omega_N^p} \left(\tfrac{1}{\mu} \underline{\underline{K}} \nabla p \cdot \underline{n} \right) \cdot \underline{\Pi} ds + \int_{\Omega} \underline{\Pi}^T q \cdot dx + \\ &\int_{\Omega} \left(\tfrac{\rho g}{\mu} \underline{\underline{K}} \cdot \nabla \underline{\Pi} (\nabla z)^T \right) dx \end{aligned} \tag{9}$$

The loose coupling approach can be obtained in different ways. One possibility is shown in Eq. (10), where the displacements are solved first taking the pressures from the previous time step. Next, pressures are updated using the new

displacements:

$$\begin{aligned} \underline{\underline{\mathrm{K}}} \cdot \underline{u}^{n+1} &= f_u + \underline{\underline{\mathrm{Q}}} \left(\underline{p}^n - \underline{p}^0 \right) \\ \underline{\underline{\mathrm{S}'}} \cdot \underline{p}^{n+1} &= \underline{\underline{\mathrm{S}''}} \cdot \underline{p}^n + \underline{f_p} \cdot \Delta t - \underline{\underline{\mathrm{Q}}}^T \left(\underline{u}^{n+1} - \underline{u}^n \right) \\ \underline{\underline{\mathrm{S}'}} &= \underline{\underline{\mathrm{S}}} + \theta \cdot \Delta t \cdot \underline{\underline{\mathrm{H}}} \\ \underline{\underline{\mathrm{S}''}} &= \underline{\underline{\mathrm{S}}} - (1 - \theta) \cdot \Delta t \cdot \underline{\underline{\mathrm{H}}} \end{aligned} \tag{10}$$

where θ is the theta-method factor, which lies between 0 and 1, and Δt is the timestep size. An iterative coupling scheme can be defined in different ways, but basically they draw from the loose coupling scheme with the addition of an internal iteration in order to update lagged quantities. For more details refer to [1]. In particular, Kim et al. [6] discuss in detail several iteratively-coupled methods for the solution of coupled flow and reservoir geomechanics for single-phase flow.

2 The Mortar Finite Element Method

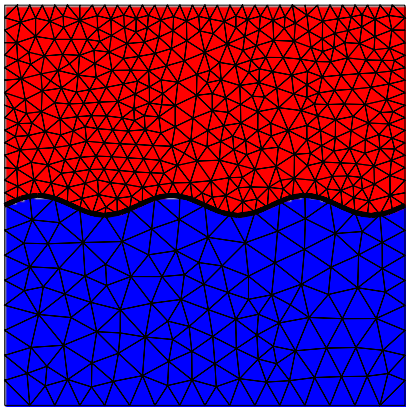

Fig. 1. Non-matching interfaces and hanging-nodes are treated properly by means of the MFEM; Ω_1 in red, while Ω_2 in blue and the interface Γ is highlighted in bold black

The main goal here is to extend MFEM to glue curve interfaces such as the one shown in Figure 1. Let MFEM be described for linear isotropic elasticity in

terms of bilinear forms, a and β defined in Eq. (11) below [16,24],

$$\begin{aligned} a\left(\underline{u},\underline{v}\right) &= \int_{\Omega} \underline{\varepsilon}(\underline{v})^T \cdot \underline{\underline{C}} \cdot \underline{\varepsilon}\left(\underline{u}\right) dx \\ l\left(\underline{v}\right) &= \int_{\partial\Omega^N} \underline{t}^T \cdot \underline{v} ds + \int_{\Omega} \underline{f}^T \cdot \underline{v} dx \\ \beta\left(\underline{u},\underline{v}\right) &= \int_{\Gamma} \left[\underline{u}\right]^T \cdot \underline{v} ds \;;\; \left[\underline{u}\right] = \left(\underline{u}^{(1)} - \underline{u}^{(2)}\right) \end{aligned} \tag{11}$$

where β stands for the gluing condition among subdomain interfaces, the jump $[\underline{u}]$ on the displacements is required to vanish in an integral or "weak"sense, thus:

$$\begin{cases} a\left(\underline{u}_h,\underline{v}_h\right) + \beta\left(\underline{v}_h,\underline{\lambda}_h\right) = l\left(\underline{v}_h\right) \\ \\ \beta\left(\underline{u}_h,\underline{\mu}_h\right) = 0 \end{cases} \tag{12}$$

in Eq. (12) $\underline{\mu}_h$ is the mortar space while $\underline{v}_h$ is the weighting space and $\underline{\lambda}_h$ is the Lagrange multiplier space, i.e. the linear combination of mortar functions and Lagrange multiplier degrees of freedom. In algebraic or matrix form, Eq. (12) can be written as:

$$\begin{bmatrix} \left[k^{(1)}\right] & [0] & \left[\beta^{(1)}\right]^T \\ [0] & \left[k^{(2)}\right] & -\left[\beta^{(2)}\right]^T \\ \left[\beta^{(1)}\right] & -\left[\beta^{(2)}\right] & [0] \end{bmatrix} \cdot \begin{bmatrix} \underline{u}^{(1)} \\ \underline{u}^{(2)} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{l}^{(1)} \\ \underline{l}^{(2)} \\ \underline{0} \end{bmatrix} \tag{13}$$

This is the so-called saddle-point problem. Notice that subdomains are only connected by means of the Lagrange multiplier $\underline{\lambda}$, if these multipliers happen to be known (it turns out that for elasticity, the multipliers are the unknown tractions on the interface), it is possible to decouple the system in Eq. (13) and then one just needs to perform subdomain solves. The rectangular matrices $\left[\beta^{(i)}\right], i = 1 \ldots 2$, are called projectors since they allow one to map to and from the mortar space. In order to compute the projector, special quadrature rules must be employed:

$$\beta_{ij}^{(k)} = \int_{u_o}^{u_n} \varphi_j^{(k)}\left(u\right) \mu_i\left(u\right) \cdot \left\|\underline{\sigma}_u\left(u\right)\right\| du \tag{14}$$

where $\varphi_j^{(k)}$ represents the global non-mortar side interpolation functions and μ_i are the mortar space basis function, while $\|\underline{\sigma}_u\|$ is the length of the tangent vector associated to the NURBS curve. The integration must be carried out on subdomain interfaces in a parametric or computational space.

3 Domain Decomposition Methods

Domain Decomposition Methods (DDM) are very efficient algorithms to compute the solution of large scale problems on parallel computers. These methods

mainly consist of splitting the global domain into several subdomains and then computing the solution on the global domain through the resolution of the subdomain problems [3,45]. There is a broad literature covering these schemes; a simple introduction is presented here for the sake of completeness. Bjrstad and Widlund [46], Bramble et al. and Marini and Quarteroni [47]considered the Dirichlet-Neumann (DN) DDM, for instance, among other authors [48].

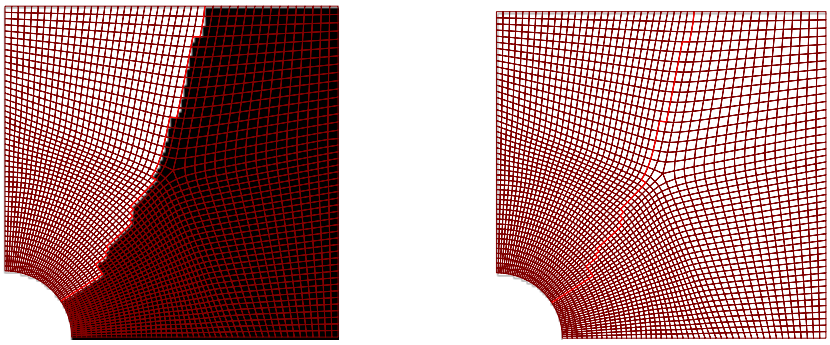

Fig. 2. Domain Decomposition schemes. The Dirichlet-Neumann (left) and the Neumann-Neumann (right) schemes are depicted

Let L be an abstract linear differential operator, which might be the Laplace operator for instance. The DN-DDM scheme consists of solving a series of problems in the right sequence, which requires a coloring tool (see Figure 2). Let us color the Dirichlet subdomains in white and the Neumann subdomains in black. Notice also that the interface between subdomains is Γ. After providing an initial guess on the primary variables on Γ, i.e. λ^k must be given, we may solve the problem on the white subdomains (Dirichlet problems), which is step 1 in Eq. (15).

$$1)\begin{cases} L{u_1}^{k+1} = f \text{ in } \Omega_1 \\ {u_1}^{k+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \\ {u_1}^{k+1} = \lambda^k \text{ on } \Gamma \end{cases} \qquad 2)\begin{cases} L{u_2}^{k+1} = f \text{ in } \Omega_2 \\ {u_2}^{k+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega \\ \partial_n {u_2}^{k+1} = \mu^{k+1} \text{ on } \Gamma \end{cases} \tag{15}$$

Let us call the primary variable "displacements"and "tractions"their gradient or normal derivative in the boundary. The tractions on the interface Γ can be computed after solving step 1 on the white subdomains, those tractions can then be passed to solve step 2 on the black subdomains, i.e. Neumann subdomains.

On these latter ones, since the tractions are being prescribed on Γ, we can solve for unknown displacements in order to provide feedback for the next iteration level. In order to improve the convergence both displacements and tractions are over-relaxed, the relaxation parameters θ^D and θ^N lie between 0 and 1 as shown in Eq. (16):

$$\begin{aligned} \mu^{k+1} &= \left(-\theta^N \cdot \partial_n {u_1}^{k+1} + \left(1-\theta^N\right) \cdot \partial_n {u_2}^k\right) \text{ on } \Gamma \\ \lambda^{k+1} &= \left(\theta^D \cdot {u_2}^{k+1} + \left(1-\theta^D\right) \cdot {u_1}^k\right) \text{ on } \Gamma \end{aligned} \tag{16}$$

It turns out that this scheme may require at least a two-entry coloring tool or even more [3]. There is a lack of parallelism in the sense that the black subdomains must wait for the others to communicate tractions. In order to mitigate this problem an initial guess for tractions can be prescribed, but this is not straightforward in most cases. An easy way to provide an initial guess for the multiplier λ^k is by doing the so-called coarse-run, which means solving the same problem in a coarser mesh and assigning λ^k on Γ by standard interpolation, using the finite element space from the coarse problem. The reader may refer to the literature [49,50] for proof of convergence and further details about this scheme. The reader may also refer to [3] for a detailed description and comments about implementation details, which are being omitted here for the sake of brevity.

4 Numerical Example

This MFEM model was implemented in the Integrated Parallel Finite Element Analysis program (IPFA), which is a C++ application that the author developed at the Center for Subsurface Modeling (CSM) at The University of Texas at Austin. The example included here was ran on a MacBook Pro laptop equipped with an Intel(R) Quad-Core(TM) i7-2720QM CPU @ 2.20GHz and 8 GB of RAM.

4.1 Two-dimensional steady state single-phase flow

This example consists of a manufactured problem where the solution is known a priori. The idea is to substitute a given pressure field in the governing equation in order to derive the source term that corresponds to that solution precisely. In strong form the problem looks like:

$$\begin{aligned} -\nabla \cdot \left(\underline{\underline{K}} \nabla p\right) &= f \text{ in } \Omega \ , \\ p &= p_0 \text{ on } \Gamma_{\mathrm{D}} = \Gamma \ , \end{aligned} \tag{17}$$

the domain is the unitary square with pure Dirichlet boundary conditions, the pressure is proposed to be [51]:

$$p\left(x,y\right) = xy \cdot \left(x-1\right) \cdot \left(y-1\right) \cdot \exp\left[-\left(x^2+y^2\right)\right] \ ; \ \underline{\underline{K}} = \underline{\underline{\mathrm{I}}} \tag{18}$$

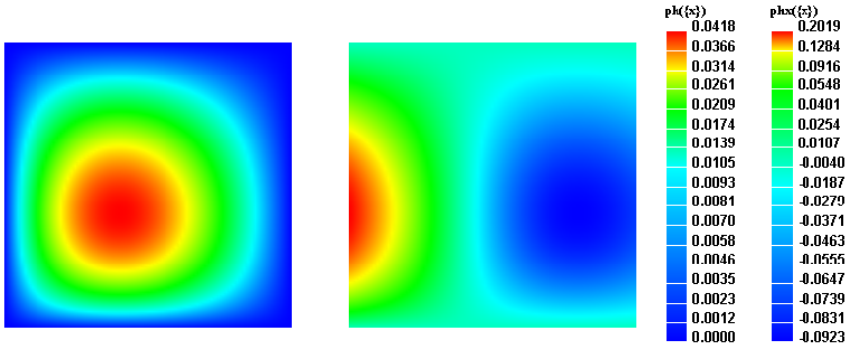

Fig. 3. Pressure contour for Eq. (18) (left) and its horizontal partial derivative (right) are shown

the pressure field is depicted in Figure 3.

Figure 4 shows the solution to problem 4.1, which consists of three-subdomains, two of them (the top and bottom ones) are triangular meshes while the one in the middle is a tensor-product quadrilateral mesh. In the top-left-corner of the figure the mesh employed is shown, the pressure is in the right-top-corner, and its horizontal derivative is in the bottom-left-corner, while the absolute error between the numerical and true solutions is in the right-bottom-corner. Table 1 depicts the number of elements and points of each mesh from top to bottom. The mortars as geometrical entities consist of two B-Splines interpolants (NURBS with all weights equal 1) constructed by interpolating a sinusoidal wave as shown. Thirty-two quadratic mortar elements per curve were employed to glue these three subdomains. A frontal direct solver was employed to solve the global saddle-point problem Eq. (13) [52]. The results depicted on Figure 4 show excellent accordance with the analytical solution, Figure 4 depicts the absolute error against the true solution, which is very small and corresponds to the error predicted by the theory. Notice that besides the displacements being matched on the interface, an excellent agreement can be also seen for the horizontal derivative as well.

Table 1. Meshes employed to solve example 4.1

Kind of mesh	Elements	Points
Triangular	1814	980
Quadrilateral	1472	1560
Triangular	7858	4090

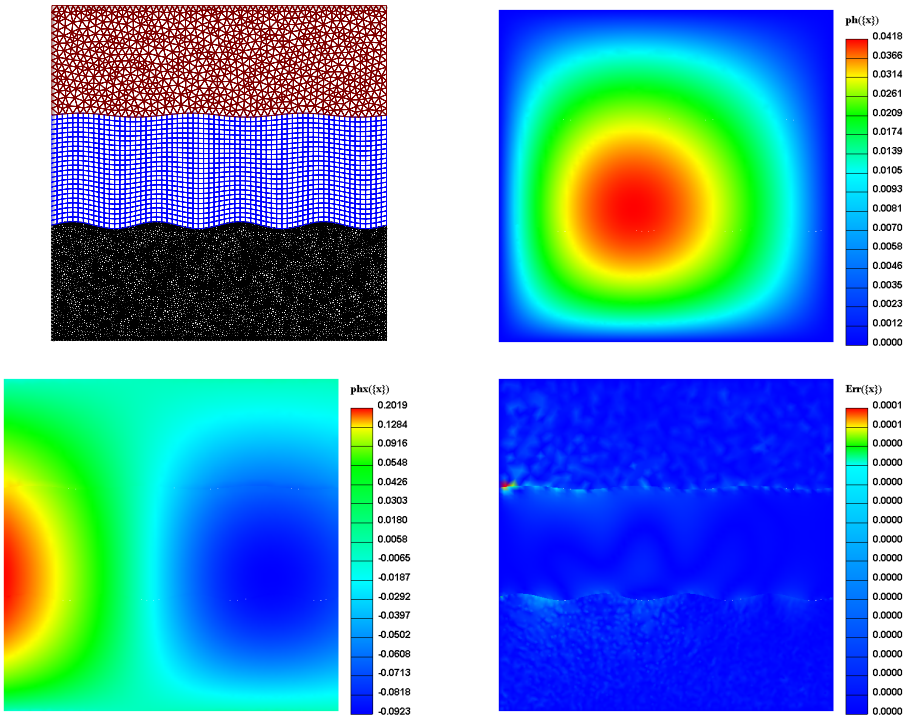

Fig. 4. The MFEM solution to problem 4.1 is depicted

Whether or not the saddle-point problem approach is used, the local problems are completely disconnected, which can be exploited to reduce the computational time. Indeed, these subdomains problems can be assembled in separated threads using a shared memory approach, i.e. multi-threading assembling. We now proceed to conduct a convergence analysis by running successively refined meshes as shown in Figure 5. We keep a refinement ratio of 2:1 between the left and right subdomains. We employ a piecewise quadratic mortar space where the number of mortar elements equals the number of coarse edges in the non-mortar sides. The curve interface in this case corresponds to a sinusoidal wave function that was interpolated with cubic B-Splines curves. The number of discrete points that were interpolated equals the number of points in the more refined non-mortar side, i.e. the left-hand-side, which guarantees that all points in the interface lie in the B-Spline curve. We recall that this is important for computing the mortar projector in a straightforward manner. We ran meshes of size 8, 16, 32, 64, 128 and 256 respectively (from the right domain point of view). The horizontal mesh resolution is consistent with the finer domain mesh size, i.e. left domain.

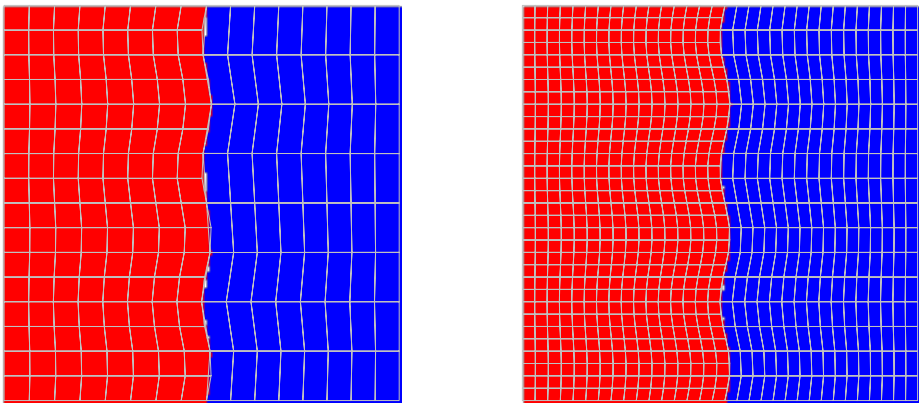

Fig. 5. Two successively refined meshes for problem 4.1 are depicted

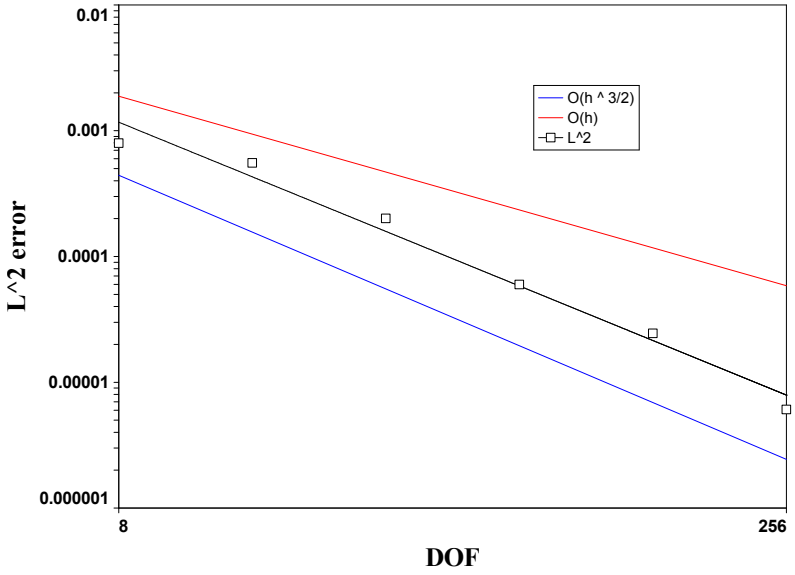

Fig. 6. The numerical L^2 convergence rate for problem 4.1 is shown

Figure 6 shows the resulting numerical convergence rate in a $log - log$ plot, the slope is 1.44143 with a coefficient of determination $R^2 = 84\%$. This agrees with the theory that predicts a rate of $\mathcal{O}(h^{3/2})$ [24,26].

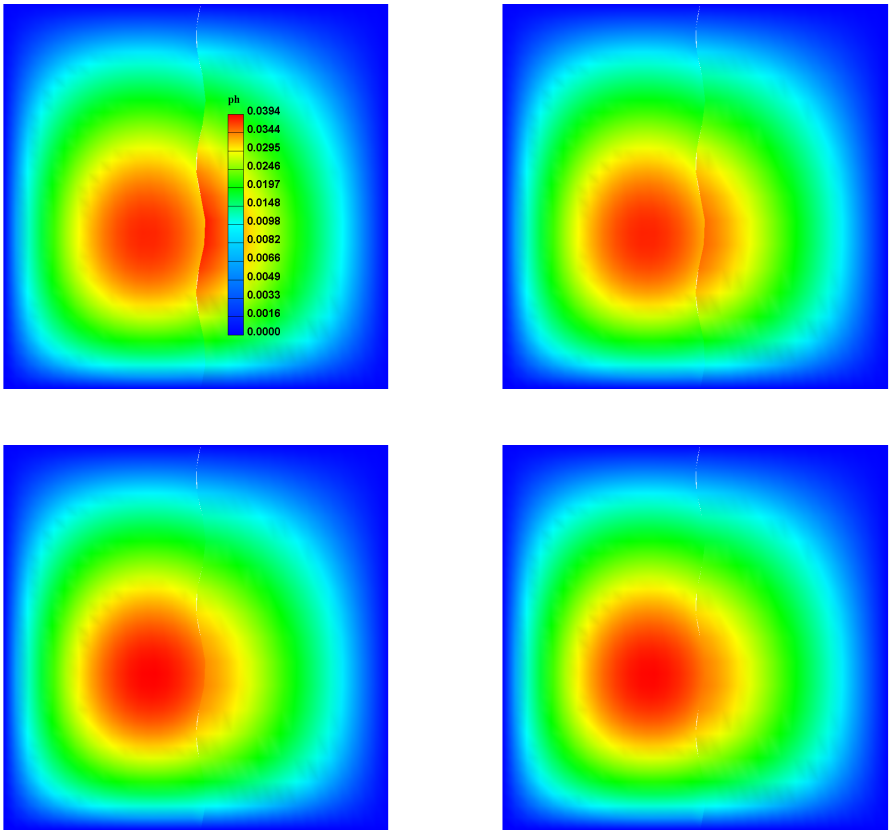

Fig. 7. Snapshots show the evolution of the DN-DDM applied to problem 4.1

Finally, Figure 7 shows pressure snapshots that correspond to four different Dirichlet-Neumann iteration levels evolving from left-to-right and top-to-bottom. We employ the mesh in the right-hand-side of Figure 5. No initial guess for pressure was provided, which explains the mismatch in the first snapshot. Notice then how the process to match up those subdomains can eliminate discrepancies in just a few iterations. The stop criterion implies that the residual in the tractions in the interface is required to drop below a given tolerance. For this problem, 6 iterations were spent to achieve a tolerance of 10^{-6}.

Conclusions

A fairly general MFEM on curved interfaces represented by means of NURBS was presented in this extended abstract. Several 2-D examples ranging from near borehole to field level compaction and subsidence computations demonstrated that the proposed scheme could handle problems of practical interest. No numerical oscillations occurred for the transient pressure field reported here. Popular DDM were employed to decouple the global saddle-point problem for elasticity, which is crucial in order to extend this methodology to realistic 3-D problems, where solving the system in Eq. (13) is not tractable at all. Some implementation details to extend the method to 3-D problems were also discussed. Forthcoming papers may address those details precisely.

Future Work

1. Extend the proposed MFEM to 3-D problems. The mortar will be geometrically represented by means of NURBS surfaces, which follow naturally from the curves discussed here. The high level steps are going to be the same but some complications may be found on computing the projector as well as the rank of the linear systems to solve for.
2. Couple IPFA to major reservoir simulators, such as black oil and compositional flow models, via loose and/or iterative coupling [10,3]. This will be the ultimate goal in order to ensure that the right physics for flow, i.e. a locally mass conservative scheme, is being taken into account.
3. Consider coupling between elasticity and plasticity in order to tackle problems on poroplasticity.

Acknowledgments

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Comparisons of Measurement Results as Constraints on Accuracies of Measuring Instruments: When Can We Determine the Accuracies from These Constraints?

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Abstract. For a measuring instrument, a usual way to find the probability distribution of its measurement errors is to compare its results with the results of measuring the same quantity with a much more accurate instrument. But what if we are interested in estimating the measurement accuracy of a state-of-the-art measuring instrument, for which no more accurate instrument is possible? In this paper, we show that while in general, such estimation is not possible; however, can uniquely determine the corresponding probability distributions if we have several state-of-the-art measuring instruments, and for one of them, the corresponding probability distribution is symmetric.

1 Formulation of the Problem

Need to determine accuracies of measurement instruments. Most information comes from measurements. Measurement results are never absolutely accurate: the measurement result $\widetilde{x}$ is, in general, different from the actual (unknown) value x of the corresponding quantity; see, e.g., [7]. To properly process data, it is therefore important to know how accurate are our measurements.

Ideally, we would like to know what are the possible values of measurement errors $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$, and how frequent are different possible values of Δx. In other words, we would like to know the probability distribution on the set of all possible values of the measurement error Δx.

How accuracies are usually determined: by using a second, much more accurate measuring instrument. One usual way to find the desired probability distribution is to have a second measuring instrument which is much more accurate than the one that we want to estimate. In this case, the measurement

error $\Delta x_2 = \widetilde{x}_2 - x$ of this second instrument is much smaller than $\Delta x = \widetilde{x} - x$ and thus, the difference $\widetilde{x} - \widetilde{x}_2 = (\widetilde{x} - x) - (\widetilde{x}_2 - x)$ between the two measurement results can serve as a good approximation to the measurement error. From the sample of such differences, we can therefore find the desired probability distribution for Δx.

What if we do not have a more accurate measuring instrument? But what if the measuring instrument whose accuracy we want to estimate is among the best? In this case, we do not have a much more accurate measuring instrument. What can we do in this case?

In such situations, we can use the fact that there usually, there are *several* measuring instrument of the type that we want to analyze. Due to measurement errors, for the same quantity, these instruments, in general, produce slightly different measurement results. It is therefore desirable to try to extract the information about measurement accuracies from the differences between these measurement results.

Two possible situations. In some cases, we have a stable manufacturing process that produces several practical identical measuring instruments, for which the probability distributions of measurement error are the same. In such cases, all we need to find is this common probability distribution.

In other cases, we cannot ignore the differences between different instruments. In this case, for each individual measuring instrument, we need to find its own probability distribution.

What is known: case of normal distribution. In many practical situations, the measurement error is caused by the joint effect of numerous independent small factors. In such situations, the Central Limit Theorem (see, e.g., [9]) implies that this distribution is close to Gaussian.

A Gaussian distribution is uniquely determined by its mean (bias) and standard deviation σ. When we only know the differences, we cannot determine the bias: it could be that all the measuring instruments have the same bias, and we will never determine that since we only see the differences. Thus, it makes sense to limit ourselves only to the *random component* of the measurement error, i.e., to the measurement error minus its mean value.

For this "re-normalized" measurement error Δx, the mean is 0. So, all we need to determine is the standard deviation σ. These standard deviations can indeed be determined; see, e.g., [4, 8].

Specifically, hen we have two identical independent measuring instruments, with normally distributed measurement errors Δx_1 and Δx_2, then the difference $\widetilde{x}_2 - \widetilde{x}_1$ is also normally distributed, with variance $V = \sigma^2 + \sigma^2 = 2\sigma^2$. Thus, once we experimentally determine the variance V of this observable difference, we can compute the desired variance σ^2 as $\sigma^2 = \dfrac{V}{2}$.

When we have several different measuring instruments, with unknown standard deviations σ_1, σ_2, σ_3, ..., then for each observable difference $\widetilde{x}_i - \widetilde{x}_j$ the variance is equal to $V_{ij} = \sigma_i^2 + \sigma_j^2$. Thus, once we experimentally determine the three variances V_{12}, V_{23}, and V_{13}, we can find the desired standard deviations

by solving the corresponding system of three equations with three unknowns: $V_{12} = \sigma_1^2 + \sigma_2^2$, $V_{23} = \sigma_2^2 + \sigma_3^2$, and $V_{13} = \sigma_1^2 + \sigma_3^2$, whose solution is:

$$\sigma_1^2 = \frac{V_{12} + V_{13} - V_{23}}{2}, \quad \sigma_2^2 = \frac{V_{12} + V_{23} - V_{13}}{2},$$

$$\sigma_3^2 = \frac{V_{13} + V_{23} - V_{12}}{2}.$$

Problem: what if distributions are not Gaussian? Empirical analysis of measuring instruments shows that only slightly more than a half of them have Gaussian measurement errors [3, 6]. What happens in the non-Gaussian case? In such cases, sometimes, we simply cannot uniquely reconstruct the corresponding distributions; see, e.g., [8]. In this paper, we explain when such a reconstruction is possible and when it is not possible.

2 Idea: Let Us Use Moments

Motivation for using moments. As we have mentioned, a Gaussian distribution with zero mean is uniquely determined by its second moment $M_2 = \sigma^2$. This means that all higher moments $M_k \stackrel{\text{def}}{=} E[(\Delta x)^k]$ are uniquely determined by the value M_2.

In general, we may have values of M_k which are different from the corresponding Gaussian values. Thus, to describe a general distribution, in addition to the second moment, we also need to describe its higher moments as well.

Moments are sufficient to uniquely describe a distribution: reminder. But even if we know all the moments, will it be sufficient to uniquely determine the corresponding probability distribution? The answer is yes, it is possible, and let us provide a simple reminder of why it is possible – and how can we reconstruct the corresponding distribution.

The usual way to represent a probability distribution of a random variable Δx is by describing its probability density function (pdf) $\rho(\Delta x)$. In many situations, it is convenient to use its *characteristic function*

$$\chi(\omega) \stackrel{\text{def}}{=} E[\exp(\mathrm{i} \cdot \omega \cdot \Delta x)],$$

where $\mathrm{i} \stackrel{\text{def}}{=} \sqrt{-1}$, i.e.,

$$\chi(\omega) = \int \rho(\Delta x) \cdot \exp(\mathrm{i} \cdot \omega \cdot \Delta x)\, d\Delta x.$$

From the mathematical viewpoint, the characteristic function is the Fourier transform of the pdf, and it is known that we can uniquely reconstruct a function from its Fourier transform (this reconstruction is known as the *inverse Fourier transform*); see, e.g., [1, 2, 5, 10].

On the other hand, if we use Taylor expansion of the exponential function

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^k}{k!} + \ldots,$$

then the characteristic functions takes the form

$$\chi(\omega) = E\left[1 + \mathrm{i} \cdot \omega \cdot \Delta x - \frac{1}{2!} \cdot \omega^2 \cdot (\Delta x)^2 + \ldots + \frac{\mathrm{i}^k}{k!} \cdot \omega^k \cdot (\Delta x)^k + \ldots\right],$$

i.e.,

$$\chi(\omega) = 1 - \frac{1}{2} \cdot \omega^2 \cdot M_2 + \ldots + \frac{\mathrm{i}^k}{k!} \cdot \omega^k \cdot M_k + \ldots$$

Thus, if we know all the moments M_k, we can uniquely reconstruct the characteristic function and thus, uniquely reconstruct the desired pdf.

Important fact: for a symmetric distribution, odd moments are zeros. In the following analysis, it is important to use the fact that for a symmetric distribution, i.e., a distribution for which $\rho(-\Delta x) = \rho(\Delta x)$, add odd moments M_{2s+1} are equal to 0:

$$M_{2s+1} = \int \rho(\Delta x) \cdot (\Delta x)^{2s+1}\, d\Delta x.$$

Indeed, if we replace Δx to $\Delta x' \stackrel{\text{def}}{=} -\Delta x$, then $d\Delta x = -d\Delta x'$, $(\Delta x)^{2s+1} = -(\Delta x')^{2s+1}$ and thus, the above integral takes the form

$$M_{2s+1} = -\int \rho(-\Delta x') \cdot (\Delta x')^{2s+1}\, d\Delta x' = -\int \rho(\Delta x') \cdot (\Delta x')^{2s+1}\, d\Delta x',$$

so $M_{2s+1} = -M_{2s+1}$ and hence, $M_{2s+1} = 0$.

3 Case When Have Several Identical Measuring Instruments

Description of the case: reminder. In this cases, we have several measuring instruments, with the same probability distribution and thus, with the same moments M_2, M_3, etc. The only available information consists of the differences $\Delta x_1 - \Delta x_2 = \widetilde{x}_1 - \widetilde{x}_2$. Based on the observations, we can determine the probability distribution for each such difference, and thus, we can determine the moments M'_k of this difference.

We would like to use these observable moments $M'_k = E[(\Delta x_1 - \Delta x_2)^k]$ to find the desired differences $M_k = E[(\Delta x)^k]$.

What is known: case of second moments. For $k = 2$, we have $M'_2 = 2M_2$ and thus, we can uniquely reconstruct the desired second moment M_2 from the observed second moment M'_2.

Natural next case: third moments. Can we similarly reconstruct the desired third moment $M_3 = E[(\Delta x)^3]$ based on the observed third moment $M_3' = E[(\Delta x_1 - \Delta x_2)^3]$?

Here,

$$(\Delta x_1 - \Delta x_2)^3 = (\Delta x_1)^3 - 3 \cdot (\Delta x_1)^2 \cdot \Delta x_2 + 3 \cdot \Delta x_1 \cdot (\Delta x_2)^2 - (\Delta x_2)^3,$$

so, due to linearity of the mean and to the fact that the measurement errors Δx_1 and Δx_2 corresponding to two measuring instruments are assumed to be independent, we conclude that

$$M_3' = E[(\Delta x_1 - \Delta x_2)^3] = E[(\Delta x_1)^3] - 3 \cdot E[(\Delta x_1)^2] \cdot E[\Delta x_2] +$$

$$3 \cdot E[\Delta x_1] \cdot E[(\Delta x_2)^2] - E[(\Delta x_2)^3].$$

In this case, $E[\Delta x_i] = 0$ and $E(\Delta x_1)^3] = E[(\Delta x_2)^3] = M_3$, so

$$M_3' = M_3 - M_3 = 0.$$

In other words, the observed third moment M_3' is always equal to 0, and thus, carries no information about M_3.

So, the only case when we can reconstruct M_3 is when we know it already. One such case is when we know that the distribution is symmetric. In turns out that in this case, we can reconstruct all the moments and thus, we can uniquely reconstruct the original probability distribution.

When the probability distribution of the measurement error is symmetric, this distribution can be uniquely determined from the observed differences. For a symmetric distribution, all odd moments are equal to 0. Thus, to uniquely determine a symmetric distribution, it is sufficient to determine all its even moments M_{2s}. Let us prove, by induction, that we can reconstruct all these even moments.

We already know that we can reconstruct M_2. Let us assume that we already know how to reconstruct the moments $M_2, \ldots, M_{2s}$. Let us show how to reconstruct the next moment $M_{2s+2} = E[(\Delta x)^{2s+2}]$. For this, we will use the observed moment $M_{2s+2}' = E[(\Delta x_1 - \Delta x_2)^{2s+2}]$. Here,

$$(\Delta x_1 - \Delta x_2)^{2s+2} = (\Delta x_1)^{2s+2} - (2s+2) \cdot (\Delta x_1)^{2s+1} \cdot \Delta x_2 +$$

$$\frac{(2s+2) \cdot (2s+1)}{1 \cdot 2} \cdot (\Delta x_1)^{2s} \cdot (\Delta x_2)^2 - \ldots +$$

$$\frac{(2s+2) \cdot (2s+1)}{1 \cdot 2} \cdot (\Delta x_1)^2 \cdot (\Delta x_2)^{2s} - (2s+2) \cdot \Delta x_1 \cdot (\Delta x_2)^{2s+1} + (\Delta x_2)^{2s+2}.$$

Thus,

$$M_{2s+2}' = E[(\Delta x_1)^{2s+2}] - (2s+2) \cdot E[(\Delta x_1)^{2s+1}] \cdot E[\Delta x_2] +$$

$$\frac{(2s+2) \cdot (2s+1)}{1 \cdot 2} \cdot E[(\Delta x_1)^{2s}] \cdot E[(\Delta x_2)^2] - \ldots +$$

$$\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot E[(\Delta x_1)^2]\cdot E[(\Delta x_2)^{2s}]-$$

$$(2s+2)\cdot E[\Delta x_1]\cdot E[(\Delta x_2)^{2s+1}]+E[(\Delta x_2)^{2s+2}],$$

i.e.,

$$M'_{2s+2}=M_{2s+2}+\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot M_{2s}\cdot M_2+\ldots+$$

$$\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot M_2\cdot M_{2s}+M_{2s+2}.$$

Thus,

$$2M_{2s+2}=M'_{2s+2}-\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot M_{2s}\cdot M_2-\ldots-$$

$$\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot M_2\cdot M_{2s}.$$

We know the value M'_{2s+2}, and we assumed that we have already shown that we can uniquely determine the moments M_2, ..., M_{2s}. Thus, we can indeed uniquely determine the moment M_{2s+2}.

Induction proves that we can indeed determine all the even moments.

4 Case When Have Several Different Measuring Instruments

Description of the case: reminder. In this case, we have several measuring instruments with, in general, different probability distributions. For each of the measuring instruments i, we want to find the corresponding moments

$$M_{k,i}=E[(\Delta x_i)^k].$$

To find these moments, we can use the observe moments

$$M'_{k,i,j}=E[(\Delta x_i-\Delta x_j)^k].$$

What is known: case of second moments. For $k=2$, we have $M'_{2,i,j}=M_{2,i}+M_{2,j}$, so we can uniquely reconstruct the desired second moments $M_{2,i}$ from the observed moments $M'_{2,i,j}$ by using the following formulas:

$$M_{2,1}=\frac{M'_{2,1,2}+M'_{2,1,3}-M'_{2,2,3}}{2},\quad M_{2,2}=\frac{M'_{2,1,2}+M'_{2,2,3}-M'_{2,1,3}}{2},$$

$$M_{2,3}=\frac{M'_{2,1,3}+M'_{2,2,3}-M'_{2,1,2}}{2}.$$

Natural next case: third moments. Can we similarly reconstruct the desired third moments $M_{3,i} = E[(\Delta x_i)^3]$ based on the observed third moments $M'_{3,i,j} = E[(\Delta x_i - \Delta x_i)^3]$?

Here,

$$(\Delta x_i - \Delta x_j)^3 = (\Delta x_i)^3 - 3 \cdot (\Delta x_i)^2 \cdot \Delta x_j + 3 \cdot \Delta x_i \cdot (\Delta x_j)^2 - (\Delta x_j)^3,$$

so, due to linearity of the mean and to the fact that the measurement errors Δx_i and Δx_j corresponding to two measuring instruments are assumed to be independent, we conclude that

$$M'_{3,i,j} = E[(\Delta x_i - \Delta x_j)^3] = E[(\Delta x_i)^3] - 3 \cdot E[(\Delta x_i)^2] \cdot E[\Delta x_j] +$$

$$3 \cdot E[\Delta x_i] \cdot E[(\Delta x_j)^2] - E[(\Delta x_j)^3].$$

In this case, $E[\Delta x_i] = E[\Delta x_j] = 0$ and $E(\Delta x_i)^3] = M_{3,i}$, so

$$M'_{3,i,j} = M_{3,i} - M_{3,j}.$$

Since we only know the differences between the their moments, we cannot uniquely reconstruct these moments $M_{3,i}$: for example, if we add a constant to all the values $M_{3,i}$, all the observed differences will not change.

So, the only case when we can reconstruct the third moments $M_{3,i}$ is when we have some information about them already. One such case is when we know that for one of the measuring instruments, the probability distribution of measurement errors is symmetric. In turns out that in this case, we can reconstruct all the moments and thus, we can uniquely reconstruct all the original probability distributions.

When the probability distribution of one of the measurement errors is symmetric, all distributions can be uniquely determined from the observed differences. Without losing generality, let us assume that the probability distribution of the measurement error is symmetric for the 1st measuring instrument. For a symmetric distribution, all odd moments are equal to 0; thus, we have $M_{2s+1,1} = 0$ for all s. Let us prove, by induction, that we can reconstruct all the moments of all the distributions.

We already know that we can reconstruct the second moments $M_{2,i}$. Let us assume that we already know how to reconstruct the moments $M_{2,i}$, ..., $M_{n,i}$. Let us show how to reconstruct the next moments $M_{n+1,i} = E[(\Delta x_i)^{n+1}]$. For this, we will use the observed moments $M'_{n+1,i,j} = E[(\Delta x_i - \Delta x_j)^{n+1}]$. We will consider two cases:

- when n is odd, i.e., $n = 2s + 1$ and $n + 2 = 2s + 2$, and
- when n is even, i.e., $n = 2s$ and $n + 1 = 2s + 1$.

First case. Let us first consider the first case. Here,

$$(\Delta x_i - \Delta x_j)^{2s+2} = (\Delta x_i)^{2s+2} - (2s + 2) \cdot (\Delta x_i)^{2s+1} \cdot \Delta x_j +$$

$$\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot(\Delta x_i)^{2s}\cdot(\Delta x_j)^2-\ldots+$$

$$\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot(\Delta x_i)^2\cdot(\Delta x_j)^{2s}-(2s+2)\cdot\Delta x_i\cdot(\Delta x_j)^{2s+1}+(\Delta x_j)^{2s+2}.$$

Thus,

$$M'_{2s+2,i,j}=E[(\Delta x_i)^{2s+2}]-(2s+2)\cdot E[(\Delta x_i)^{2s+1}]\cdot E[\Delta x_j]+$$

$$\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot E[(\Delta x_i)^{2s}]\cdot E[(\Delta x_j)^2]-\ldots+$$

$$\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot E[(\Delta x_i)^2]\cdot E[(\Delta x_j)^{2s}]-$$

$$(2s+2)\cdot E[\Delta x_i]\cdot E[(\Delta x_j)^{2s+1}]+E[(\Delta x_j)^{2s+2}],$$

i.e.,

$$M'_{2s+2,i,j}=M_{2s+2,i}+\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot M_{2s,i}\cdot M_{2,j}+\ldots+$$

$$\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot M_{2,i}\cdot M_{2s,j}+M_{2s+2,j}.$$

Thus,

$$M_{2s+2,i}+M_{2s+2,j}=s_{ij}\stackrel{\text{def}}{=}M'_{2s+2,i,j}-\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot M_{2s,i}\cdot M_{2,j}-\ldots-$$

$$\frac{(2s+2)\cdot(2s+1)}{1\cdot 2}\cdot M_{2_i}\cdot M_{2s,j}.$$

We know the value $M'_{2s+2,i,j}$, and we assumed that we have already shown that we can uniquely determine the moments $M_{2,i},\ldots,M_{2s+1,i}$. Thus, we can indeed uniquely determine the values $s_{ij}=M_{2s+2,i}+M_{2s+2,j}$.

Based on these values, we can uniquely reconstruct the moments $M_{n+1,i}=M_{2s+2,i}$ as follows:

$$M_{2s+2,1}=\frac{s_{12}+s_{13}-s_{23}}{2},\quad M_{2s+2,2}=\frac{s_{12}+s_{23}-s_{13}}{2},$$

$$M_{2s+2,3}=\frac{s_{13}+s_{23}-s_{12}}{2}.$$

Second case. Let us now consider the second case, when $n=2s$ and $n+1=2s+1$. Since we assumed that for the first measuring instrument, the probability distribution is symmetric, we get $M_{2s+1,1}=E[(\Delta x_1)^{2s+1}]=0$.

For every $i\neq 1$, we have

$$(\Delta x_i-\Delta x_1)^{2s+1}=(\Delta x_i)^{2s+1}-(2s+2)\cdot(\Delta x_i)^{2s}\cdot\Delta x_1+$$

$$\frac{(2s+1)\cdot 2s}{1\cdot 2}\cdot(\Delta x_i)^{2s-1}\cdot(\Delta x_1)^2-\ldots+$$

$$\frac{(2s+1)\cdot 2s}{1\cdot 2}\cdot(\Delta x_i)^2\cdot(\Delta x_1)^{2s-1}-(2s+1)\cdot\Delta x_i\cdot(\Delta x_1)^{2s}+(\Delta x_1)^{2s+1}.$$

Thus,

$$M'_{2s+1,i,1}=E[(\Delta x_i)^{2s+1}]-(2s+2)\cdot E[(\Delta x_i)^{2s+1}]\cdot E[\Delta x_1]+$$
$$\frac{(2s+1)\cdot 2s}{1\cdot 2}\cdot E[(\Delta x_i)^{2s-1}]\cdot E[(\Delta x_1)^2]-\ldots+$$
$$\frac{(2s+1)\cdot 2s}{1\cdot 2}\cdot E[(\Delta x_i)^2]\cdot E[(\Delta x_1)^{2s-1}]-$$
$$(2s+1)\cdot E[\Delta x_i]\cdot E[(\Delta x_1)^{2s}]+E[(\Delta x_1)^{2s+1}],$$

i.e.,

$$M'_{2s+1,i,1}=M_{2s+1,i}+\frac{(2s+1)\cdot 2s}{1\cdot 2}\cdot M_{2s-1,i}\cdot M_{2,1}+\ldots+$$

Thus,

$$M_{2s+1,i}=M'_{2s+1,i,1}-\frac{(2s+1)\cdot 2s}{1\cdot 2}\cdot M_{2s,i}\cdot M_{2,1}-\ldots.$$

We know the value $M'_{2s+1,i,1}$, and we assumed that we have already shown that we can uniquely determine the moments $M_{2,i}$, ..., $M_{2s,i}$. Thus, we can indeed uniquely determine the moments $M_{n+1,i}=M_{2s+1,i}$.

Conclusion. In both cases, the induction step is proven, so induction proves that we can indeed determine all the moments of all the distributions.

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Consistent Conjectures Are Optimal Nash Strategies in the Meta-Game

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Abstract. In this paper, we investigate the properties of consistent conjectural variations equilibrium (CCVE) developed for a single-commodity oligopoly. Although, in general, the consistent conjectures are distinct from those of Cournot-Nash, we establish the following remarkable fact. Define a meta-game as such where the players are the same agents as in the original oligopoly but now using the conjectures as their strategies. Then the consistent conjectures of the original oligopoly game provide for the Cournot-Nash optimal strategies for the meta-game.

Keywords: Game theory; consistent conjectural variations equilibrium (CVE); meta-game; optimal Cournot-Nash strategies

1 Introduction

Conjectural variations equilibrium (CVE) was introduced quite long ago as another possible solution concept in static games (*cf.*, [1]–[2]). According to this concept, agents behave as follows: each agent chooses his/her most favorable action taking into account that every rival's strategy is a *conjectured function* of his/her own strategy.

In monograph [3], a new concept of conjectural variations equilibrium (CVE) was introduced and investigated, in which the conjectural variations (represented

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via the so called *influence coefficients* of each agent) provided a new equilibrium paradigm distinct from the Cournot-Nash equilibrium.

The detailed story of the highs and lows of the CVE concept is described in [4]. The main obstacle on the way of admitting this concept is the difficulty of checking its consistency. The *consistency* (or, sometimes, "rationality") of the equilibrium is defined as the coincidence between the conjectural best response of each agent and the conjectured reaction function of the same.

To cope with a conceptual difficulty arising in many-player models, Bulavsky proposed in 1997 (*cf.*, [5]) a completely new approach applied later to mixed oligopoly models [4]. Consider a group of n producers ($n \geq 2$) of a homogeneous good with the cost functions $f_i(q_i)$, $i = 1, \ldots, n$, where $q_i \geq 0$ is the output by producer i. Consumers' demand is described by a demand function $G = G(p)$, whose argument p is the market clearing price. Active demand D is nonnegative and does not depend upon the price. The equilibrium between the demand and supply for a given price p is guaranteed by the following balance equality

$$\sum_{i=1}^{n} q_i = G(p) + D. \tag{1}$$

Every producer $i = 1, \ldots, n$, chooses his/her output volume $q_i \geq 0$ so as to maximize his/her profit function

$$\pi_i(p, q_i) = p \cdot q_i - f_i(q_i). \tag{2}$$

Now we postulate that the agents (producers) suppose that their choice of production volumes may affect the price value p. The latter assumption could be defined by a conjectured dependence of the price p upon the output values q_i. If so, the first order maximum condition to describe the equilibrium would have the form:

$$\frac{\partial \pi_i}{\partial q_i} = p + q_i \cdot \frac{\partial p}{\partial q_i} - f_i'(q_i) \begin{cases} = 0, & \text{if } q_i > 0; \\ \leq 0, & \text{if } q_i = 0, \end{cases} \quad \text{for} \quad i = 1, \ldots, n. \tag{3}$$

Thus, we see that to describe the agent's behavior, we need evaluate the behavior of the derivative $\partial p / \partial q_i = -v_i$ rather than the functional dependence of p upon q_i. Then the optimality condition (3) is reduced to

$$\begin{cases} p = v_i q_i + b_i + a_i q_i, & \text{if } q_i > 0; \\ p \leq b_i, & \text{if } q_i = 0. \end{cases} \tag{4}$$

Definition 1. *A collection $(p, q_1, \ldots, q_n)$ is called an* exterior equilibrium state *for given influence coefficients $(v_1, \ldots, v_n)$, if the market is balanced, i.e., equality (1) holds, and for each i the maximum conditions (4) are valid.*

We assume the following properties of the model's data.

A1 *The demand function $G = G(p) \geq 0$ is defined for $p \in (0, +\infty)$, being non-increasing and continuously differentiable.*

The production costs are assumed to be (strictly) convex quadratic functions:

A2 *For each i, the cost function f_i is quadratic and $f_i(0) = 0$, i.e.,*

$$f_i(q_i) = (1/2)a_i q_i^2 + b_i q_i, \quad \text{where} \quad a_i > 0, b_i \geq 0, \ i = 1, \ldots, n. \tag{5}$$

From now on, we are going to consider only the case when the set of really producing participants is fixed (i.e., it *doesn't* depend upon the values v_i of the influence coefficients). To guarantee this feature, we make the assumption listed below.

A3 *For the price value $p_0 = \max\limits_{1 \leq j \leq n} b_j$, the following estimate holds:*

$$\sum_{i=1}^{n} \frac{p_0 - b_i}{a_i} < G(p_0). \tag{6}$$

Now we establish the following existence result.

Theorem 4. *Under assumptions A1, A2, and A3, for any $D \geq 0$, $v_i \geq 0$, $i = 1, \ldots, n$, there exists uniquely an exterior equilibrium $(p, q_1, \ldots, q_n)$ depending continuously upon the parameters $(D, v_1, \ldots, v_n)$. The equilibrium price $p = p(D, v_1, \ldots, v_n)$ as a function of these parameters is differentiable with respect to D and v_i, $i = 1, \ldots, n$. Moreover, $p(D, v_1, \ldots, v_n) > p_0$, and*

$$\frac{\partial p}{\partial D} = \frac{1}{\sum\limits_{i=1}^{n} \dfrac{1}{v_i + a_i} - G'(p)}. \tag{7}$$

Now having formula (7) in mind and following the ideas of [2], we introduce the following

Consistency Criterion.

At an exterior equilibrium $(p, q_1, \ldots, q_n)$, the influence coefficients v_k, $k = 1, \ldots, n$, are referred to as *consistent* if the equalities below hold:

$$v_k = \frac{1}{\sum\limits_{i=1, i \neq k}^{n} \dfrac{1}{v_i + a_i} - G'(p)}, \quad k = 1, \ldots, n. \tag{8}$$

Now we are in a position to define the concept of an interior equilibrium.

Definition 2. *A collection $(p, q_1, \ldots, q_n, v_1, \ldots, v_n)$ is called an* interior equilibrium state*, if for the considered influence coefficients, the collection $(p, q_1, \ldots, q_n)$ is an exterior equilibrium state, and the consistency criterion is valid for all $k = 1, \ldots, n$.*

The interior equilibrium existence result is as follows:

Theorem 5. *Under assumptions A1, A2, and A3, there exists an interior equilibrium state.*

Theorem 4 allows us to define the following game $\Gamma = (N, V, \Pi, D)$, which will be called the *meta-game*. Here, D is a (fixed) value of the active demand, $N = \{1, \ldots, n\}$ is the set of the same players as in the above-described model, $V = R^n_+$ represents the set of possible strategies, i.e., vectors of conjectures $v = (v_1, \ldots, v_n) \in R^n_+$ accepted by the players, and finally, $\Pi = \Pi(v) = (\pi_1, \ldots, \pi_n)$ is the collection of the payoff values defined (uniquely by Theorem 4) by the strategy vector v.

Now the main result of this paper is as follows. In general, the Cournot conjectures are not consistent in our single commodity market model. In other words, the Cournot conjectures $v_i = -p'(G)$ usually do not satisfy the (nonlinear) consistency system (8). However, in the meta-game introduced above, the consistent conjectures , determined by (8) provide the Cournot-Nash equilibrium.

Theorem 6. *Suppose that assumptions A1, A2, and A3 hold. Then any Cournot-Nash equilibrium in the meta-game $\Gamma = (N, V, \Pi, D)$ generates a consistent (interior) equilibrium in the original oligopoly. Conversely, every interior (consistent) equilibrium in the original oligopoly is Cournot-Nash equilibrium in the meta-game $\Gamma = (N, V, \Pi, D)$.*

Since the meta-game strategies set $V = R^n_+$ is unbounded, the existence of at least one Cournot-Nash equilibrium state in this game is by no means easy to check. The following three results (under some extra assumptions) guarantee that the existence of interior equilibrium in the original oligopoly imply the existence of Nash equilibrium in the meta-game.

Theorem 7. *In addition to assumptions A1, A2, and A3, suppose that the demand function is affine, that is,*

$$G(p) := \begin{cases} -Kp + T, & \text{if } 0 \le p \le \dfrac{T}{K}; \\ 0, & \text{if } p > \dfrac{T}{K}; \end{cases} \tag{9}$$

here, $K > 0, T > 0$. In this case, the consistency criterion for the original oligopoly is the necessary and sufficient condition for the collection of influence conjectures $v = (v_1, \ldots, v_n)$ to be Cournot-Nash equilibrium in the meta-game.

Theorem 8. *Let the assumptions of Theorem 7 be a bit relaxed for the demand function: Instead of (9), suppose that the function G is concave. In addition, if $n = 2$ (duopoly), there exists $\varepsilon > 0$ such that $G'(p) \le -\varepsilon$ for all $p \ge 0$. Then the consistency criterion for the original oligopoly is the necessary and sufficient condition for the collection of influence conjectures $v = (v_1, \ldots, v_n)$ to be Cournot-Nash equilibrium in the meta-game.*

Since the concavity of the demand function may be a too restrictive requirement, the next theorem relaxes it even more by replacing it with the Lipschitz continuity of the derivative $G'(p)$.

Theorem 9. *Suppose that apart from assumptions A1, A3, and A**??**, the regular demand function's derivative is Lipschitz continuous. In more detail, for $n \geq 3$ assume that for any p_1, p_2 the following inequality holds:*

$$|G'(p_1) - G'(p_2)| \leq \frac{1}{2s^2 G(p_0)} |p_1 - p_2|, \tag{10}$$

where $s = \max\{a_1, \ldots, a_n\}$, and the price p_0 is defined in assumption A3. Next, if $n = 2$ (duopoly), we again suppose that there exists $\varepsilon > 0$ such that $G'(p) \leq -\varepsilon$ for all $p \geq 0$, and the Lipschitz continuity of the demand function is described in the form:

$$|G'(p_1) - G'(p_2)| \leq \frac{2}{\left(\dfrac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3s\right)^2 G(p_0)} |p_1 - p_2|, \quad \forall p_1, p_2. \tag{11}$$

Then the consistency criterion for the original oligopoly is the necessary and sufficient condition for the collection of influence conjectures $v = (v_1, \ldots, v_n)$ to be Cournot-Nash equilibrium in the meta-game.

To resume, the paper presents a justification of the concept of consistent conjectures and thus contributes to a better understanding of the nature of conjectural variations equilibrium (CVE). In this paper, we considered an upper level game, in which not the supply volumes q_i but the conjectures (influence coefficients) v_i serve as the players' strategies instead. The remarkable fact we have demonstrated is the following: in the upper level game, the consistent (for the original game) conjectures v_i^* provide for the optimal Cournot-Nash strategies. In other words, if each player i assumes that the other agents stick to their consistent conjectures v_j^*, $j \neq i$, then his/her consistent conjecture v_i^* is optimal for player i, too. The latter means that the vector of conjectures provides the classical Cournot-Nash equilibrium in the upper level game.

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