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CoProD
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THE UNIVERSITY OF TEXAS AT EL PASO

Organizers: Dr. Martine Ceberio and Dr. Vladik Kreinovich

Welcome to COPROD'2017
10th Annual Conference of Constraints Programming and Decision Making COPROD'2017
El Paso, Texas, USA - November 3, 2017

8:30 AM	Registration and coffee CCSB G.0208
9:20 AM	Presentation 1: **Michelle Afravi and Vladik Kreinovich**, Fuzzy Systems Are Universal Approximators for Random Dependencies: A Simplified Proof
9:50 AM	Presentation 2: **Juan Carlos Figueroa**, On the representation of d_p metrics as fuzzy sets
10:20 AM	Presentation 3: **Gerardo Muela**, Why Decimal System and Binary System Are the Most Widely Used: A Possible Explanation
10:20 AM	 Coffee break 
11:00 AM	Presentation 4: **Olga Kosheleva, Vladik Kreinovich, and Martine Ceberio**, Attraction-Repulsion Forces Between Biological Cells: A Theoretical Explanation of Empirical Formulas
11:30 AM	Presentation 5: **Andrzej Pownuk and Vladik Kreinovich,** Why Unexpectedly Positive Experiences Make Decision Makers More Optimistic: An Explanation
12:00 AM	 Lunch 
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2:20 PM	Presentation 6: **Angel F. Garcia Contreras, Martine Ceberio, and Vladik Kreinovich**, Why Convex Optimization Is Ubiquitous and Why Pessimism Is Widely Spread
2:50 PM	CHANGE OF ROOM: SHORT BREAK CCSB 1.0202
3:10 PM	Presentation 7: **Andrezj Pownuk and Vladik Kreinovich**, Which Value $\tilde{x}$ Best Represents a Sample $x_1, \cdots, x_n$: Utility-Based Approach Under Interval Uncertainty
3:40 PM	Presentation 8: **Angel F. Garcia Contreras, Martine Ceberio, and Vladik Kreinovich** Plans Are Worthless but Planning Is Everything: A Theoretical Explanation of Eisenhower's Observation
4:10 PM	Short Presentation 1: **Leobardo Valera, and Martine Ceberio**, Reliable Evaluation of the L_2-Norm of a Stable Linear Filter Using Interval Constraints Solving Techniques
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5:20 PM	Short Presentation 3: **Leobardo Valera, Jesus Padilla, and Martine Ceberio**, Reliable-Eig: An Algorithm to Compute the Spectrum of a Non-Defective Matrix Using Interval Constraints Solving Techniques

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Special Talk

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Alfredo Vaccaro was born in Vico Equense (NA) on June, 5 1974.

He got the MSc. degree cum laude and commendation in Electronic Engineering from the University of Salerno discussing a thesis titled "High performance Speed Tracking of PDMC Motor Using Neuro-Fuzzy Control", and the PhD in Electrical and Computer Engineering from University of Waterloo, Ontario, Canada. An extract of his MSc. thesis was presented at the 5th UK Workshop on Fuzzy Systems, recent Advances in and Practical Applications of Fuzzy Systems, Sheffield, 26-27 May 1998. From September 1997 to February 1998 he joined the research group of Intelligent Systems and Control at the Department of Automatic control and System Engineering, University of Sheffield (UK), where he developed the experimental activities of his MSc thesis. From 1998 to 2002 he attained a research fellowship at the Power System Group of University of Salerno – Department of Electrical and Information Engineering (DIIIE) for the "Study of innovation technologies for remote vehicle management in public transportation systems" ("Studio di tecnologie innovative per la teleregolazione dei veicoli nei sistemi di trasporto pubblico locali"). From March 2002 to October 2014 he has been Assistant Professor of Electric Power Systems at the Department of Engineering, Faculty of Engineering of University of Sannio. From June 2008 to June 2012 he chaired the Research & Development Committee of the Opera21 Group SpA in the field of Advanced Information and Communications Technologies for Smart Grids From January 2009 to December 2014 he was the Task Leader of the strategic scientific initiatives of the Research Consortium on Agent Systems in the field of Smart Energy Networks. From February 2011 to December 2013 he was the Scientific Director of the bureau of the Research Centre on Pure and Applied Mathematic at the Department of Engineering, University of Sannio. On October 2014 he obtained the National Scientific Qualification of Full Professor in Electrical Energy Engineering. Since November 2014 he is Associate Professor of Electric Power Systems at the Department of Engineering, Faculty of Engineering of University of Sannio.

The Role of Affine Arithmetic in Robust Optimal Power Flow Analysis

Alfredo Vaccaro

University of Sannio, Department of Engineering, Piazza Roma 21, 82100 Benevento
Italy
vaccaro@unisannio.it

Optimal power system operation requires intensive numerical analysis to study and improve system security and reliability. In this context, power system operators need to understand and reduce the impact of system uncertainties. To address this issue, Optimal Power Flow (OPF) analys is one of the most important tool, since it represents the mathematical foundation of many power engineering applications such as state estimation, network optimization, unit commitment, voltage control, generation dispatch, and market studies.

For the most common formalization of the OPF problem, all input data are specified using deterministic variables resulting either from a snapshot of the system or defined by the analyst based on several assumptions about the system under study (e.g. expected/desired generation/load profiles). This approach provides OPF solutions for a single system state that is deemed representative of the limited set of system conditions corresponding to the data assumptions. Thus, when the input conditions are uncertain, numerous scenarios need to be analyzed. These uncertainties are due to several internal and external sources in power systems. The most relevant uncertainties are related to the complex dynamics of the active and reactive power supply and demand, which may vary due to, for example:

- the variable nature of generation patterns due to competition [1];
- the increasing number of smaller geographically dispersed generators that could sensibly affect power transactions [1];
- the difficulties arising in predicting and modeling market operators behavior, governed mainly by unpredictable economic dynamics, which introduce considerable uncertainty in short-term power system operation; and
- the high penetration of generation units powered by non-dispatchable renewable energy sources that induce considerable uncertainty in power systems operation [2].

Since uncertainties could affect the OPF solution to a considerable extent, reliable solution paradigms, incorporating the effect of data uncertainties, are required. Such algorithms could allow analysts to estimate both the data tolerance (i.e. uncertainties characterization) and the solution tolerance (i.e. uncertainty propagation assessment), providing, therefore, insight into the level of confidence of OPF solutions. Furthermore, these methodologies could effectively support sensitivity analysis of large parameters variations to estimate the rate of change in the solution with respect to changes in input data.

To address the aforementioned problem, this paper analyzes novel solution methodologies based on the use of Affine Arithmetic, which is an enhanced model for self-validated numerical analysis in which the quantities of interest are represented as affine combinations of certain primitive variables representing the sources of uncertainty in the data or approximations made during computations. Compared to existing solution paradigms, this formulation presents greater flexibility, as it allows to find partial solutions and inclusion of multiple equality and inequality constraints, and reduce the approximation errors to obtain better OPF solution enclosures.

Detailed numerical results obtained on a real case study are presented and discussed, demonstrating the effectiveness of the proposed methodologies, especially in comparison to more traditional techniques.

1 Motivations

Conventional methodologies available in the literature propose the use of sampling, analytical and approximate methods for OPF analysis [3, 4], accounting for the variability and stochastic nature of the input data used. A critical review of the most relevant papers proposing these solution methodologies is presented in the following subsections.

1.1 Sampling Methods

Uncertainty propagation studies based on sampling based methods, such as Monte Carlo, require several model runs that sample various combinations of input values. In particular, the most popular Monte Carlo based algorithm adopted to solve OPF problems is simple random sampling, in which a large number of samples are randomly generated from the probability distribution functions of the input uncertain variables. Although this technique can provide highly accurate results, it has the drawback of requiring high computation resources needed for the large number of repeated OPF solutions [5]. This hinders the application of this solution algorithm, especially for large scale power system analysis, where the number of simulations may be rather large and the needed computational resources could be prohibitively expensive [6].

The need to reduce the computational costs of Monte Carlo simulations, has stimulated the research for improved sampling techniques aimed at reducing the number of model runs, at the cost of accepting some level of risk. For example, in [7], an efficient Monte Carlo method integrating Latin hypercube sampling and Cholesky decomposition is proposed to solve PF problems. In [8], the unceratin PF problem with statistically correlated input random variables is solved by a hybrid solution algorithm based on deterministic annealing expectation maximization algorithm and Markov chain Monte Carlo. An extended Latin hypercube sampling algorithm aimed at solving PF problems in the presence of correlated wind generators is proposed in [9]. In [10], the uncertain OPF problem is formulated as a chance-constrained programming model, and the stochastic

features of its solutions are obtained by combining Monte Carlo based simulations with deterministic optimisation models.

Although the application of the aforementioned techniques allow to lower the computational burden of sampling-based approaches, these reduce the accuracy of the estimation of uncertainty regions of PF and OPF solutions. Therefore, the dichotomy between accuracy and computational efficiency is still an open problem that requires further investigation.

1.2 Analytical Methods

Analytical methods are computationally more effective, but they require some mathematical assumptions in order to simplify the problem and obtain an effective characterization of the output random variables [11]. These assumptions are typically based on model multi-linearization [12], convolution techniques, and fast Fourier transforms [13]. For example, the cumulant method has been applied to solve the probabilistic OPF problem in [11]; the performance of this method is enhanced by combining it with the Gram-Charlier expansion in [14], and by integrating the Von Mises functions in [15], to handle discrete distributions.

Analytical techniques present various shortcomings, as discussed in [16–18], such as the need to assume statistical independence of the input data, and the problems associated with accurately identifying probability distributions for some input data. This is a problem for PF and OPF analysis, since it is not always feasible to translate imprecise knowledge into probability distributions, as in the case of power generated by wind or photovoltaic generators, due to the inherently qualitative knowledge of the phenomena and the lack of sufficient data to estimate the required probability density distributions. To address this issue, the assumptions of normality and statistical independence of the input variables are often made, but experimental results show that these assumptions are often not supported by empirical evidence. These drawbacks may limit the usefulness of analytical methods in practical applications, especially for the study of large-scale power networks.

1.3 Approximate Methods

In order to overcome some of the aforementioned limitations of sampling and analytical methods, the use of approximate methods, such as the first-order second-moment method and point estimate methods, have been proposed in the literature [19]. Rather than computing the exact OPF solution, these methods aim at approximating the statistical proprieties of the output random variables by means of a probability distribution fitting algorithm. In particular, the application of the first-order second-moment method allows to compute the first two moments of the OPF solution by propagating the moments of the input variables by the Taylor series expansion of the model equations [20].

Approximate solution methods present several shortcomings. In particular, two-point estimate methods are not suitable to solve large scale problems, since

they typically do not provide acceptable results in the presence of a large number of input random variables. Moreover, the identification of the most effective scheme that should be adopted to select the number of estimated points is still an open problem that requires further investigations [21]; this is a critical issue, since a limited number of estimated points does not allow for an accurate and reliable exploration of the solution space, especially for input uncertainties characterized by relatively large standard deviations, such as in the case of lognormal or exponential distributions [21]. On the other hand, an increased number of estimated points reduces the computational benefits deriving by the application of point estimated methods, which could degenerate into a standard Monte Carlo solution approach.

1.4 Non-Probabilistic Methods

Recent research has enriched the spectrum of available techniques to deal with uncertainty in OPF by proposing self-validated computing for uncertainty representation in OPF analysis. The main advantage of self-validated computation is that the algorithm itself keeps track of the accuracy of the computed quantities, as part of the process of computing them, without requiring information about the type of uncertainty in the parameters [22]. The simplest and most popular of these models is Interval Mathematics (IM), which allows for numerical computation where each quantity is represented by an interval of floating point numbers without a probability structure [23]. Such intervals are added, subtracted, and/or multiplied in such a way that each computed interval is guaranteed to contain the unknown value of the quantity it represents.

The application of "standard" IM, referred here as interval arithmetic (IA), to PF analysis has been investigated by various authors [18, 17, 24, 25]. However, the adoption of this solution technique present many drawbacks derived mainly by the so called "dependency problem" and "wrapping effect" [22, 26]; as a consequence, the solution provided by an IA method for PF solution is not always as informative as expected. Thus in [27], we showed that the use of IA for the solution of power flow equations may easily yield aberrant solutions, due to the fact that the IA formalism is unable to represent the correlations that the power flow equations establishes between the power systems state variables; as a consequence, at each algorithm step spurious values are added to the solutions, which could converge to large domains that include the correct solution. This phenomenon is well known in the simulation of qualitative systems [28, 29], and requires the adoption of specific techniques such as the Interval Gauss elimination, the Krawczyk's method, and the Interval Gauss Seidel iteration procedure. Therefore, the application of these paradigms in the PF solution process leads to realistic solution bounds only for certain special classes of matrices (e.g. M-matrices, H-matrices, diagonally dominant matrices, tri-diagonal matrices) [30]; furthermore, to guarantee convergence, it is necessary to preconditioning the linear PF equations by an M-matrix [31]. These techniques make the application of IA to PF analysis complex and time consuming.

1.5 Affine Arithmetic-based Methods

To overcome the aforementioned limitations in IA, in [27], we proposed the employment of a more effective self validated paradigm based on Affine Arithmetic (AA) to represent the uncertainties of the PF state variables, which is one of the topics of the present thesis. In this approach, each state variable can be expressed by a first degree polynomial composed by a central value, i.e. the nameplate value, and a number of partial deviations that represent the correlation among various variables. The adoption of AA for uncertainty representation allows expressing the power flow equations in a more convenient formalism, so that a reliable estimation of the PF solution hull can be computed taking into account the parameter uncertainty inter-dependencies, as well as the diversity of uncertainty sources. The main advantage of this solution strategy is that it requires neither derivative computations nor interval systems, being thus suitable in principle for large scale power flow studies, where robust and computationally efficient solution algorithms are required. These benefits have been confirmed in [32] and in [33], which allows to determine operating margins for thermal generators in systems with uncertain parameters, by representing all the state and control variables with affine forms accounting for forecast, model error, and other sources of uncertainty, without the need to assume a probability density function. These methodologies have been recently recognized as one of the most promising alternative for stochastic information management in bulk generation and transmission systems for smart grids [34].

Based on our own work reported in [27], several papers have explored the application of AA-based computing in power system analysis. In particular, in [35] the state estimation problem in the presence of mixed phasor and conventional power measurements has been addressed, considering the effect of network parameters uncertainty by an iterative weight least square algorithm based on IA and AA processing. In [36], an AA-based model of the uncertain OPF problem is proposed, using complementarity conditions to properly represent generator bus voltage controls, including reactive power limits and voltage recovery; the model is then used to obtain operational intervals for the PF variables considering active and reactive power demand uncertainties. In [37], a non-iterative solution scheme based on AA is proposed to estimate the bounds of the uncertain PF solutions by solving an uncertain PF problem, which is formalized by an interval power flow problem and solved by quadratic programming optimization models.

The benefits deriving from the application of AA-based computing to power system planning and operation in the presence of data uncertainty have been assessed in [38], which confirms that AA represents a fast and reliable computing paradigm that allows planners and operators to cope with high levels of renewable energy penetration, electric vehicle load integration, and other uncertain sources. Moreover, as confirmed in [39, 40, 32, 33], AA allows the analyst to narrow the gap between the upper and lower bounds of the OPF solutions, avoiding the overestimation of bounds resulting from correlation of variables in IA.

Although the aforementioned papers offer considerable insight on the role that AA may play in power systems analysis, several open problems still remain unsolved, particularly:

- Further exploration of the application of AA-based techniques to uncertain OPF analysis.
- Rigorous methodologies aimed at selecting the noise symbols of the affine forms representing the power system state variables.
- More efficient paradigms aimed at reducing the overestimation errors of AA-based PF and OPF problems.

2 Paper Contributions

Based on the above literature review, the following are the main paper objectives:

1. Demonstrate with a realistic test system that the use IA in PF and OPF analysis leads to over-pessimistic estimation of the solution hull, which are not useful in most practical applications due to the inability of IA to keep track of correlations between the power systems state variables, and analyze the employment of AA to represent the uncertainties of the power systems state variables. The adoption of AA for uncertainty representation will allow to express the OPF models in a more convenient formalism compared to the traditional and widely used linearization frequently used in interval Newton methods.
2. Present and thoroughly test a novel solution methodology based on AA for OPF studies with data uncertainties. By using the proposed methodology, a reliable estimation of the OPF solutions hull will be computed, taking into account the parameter uncertainty inter-dependencies as well as the diversity of uncertainty sources. The main advantage of this solution strategy is that it doe not require the solution of interval systems of equations, being thus suitable in principle for large scale OPF studies where robust and computationally efficient solution algorithms are required.

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Long Presentations

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Fuzzy Systems Are Universal Approximators for Random Dependencies: A Simplified Proof

Mahdokht Afravi and Vladik Kreinovich

Department of Computer Science, University of Texas at El Paso
El Paso, TX 79968, USA
mafravi@miners.utep.edu, vladik@utep.edu

Abstract. In many real-life situations, we do not know the actual dependence $y = f(x_1, \ldots, x_n)$ between the physical quantities x_i and y, we only know expert rules describing this dependence. These rules are often described by using imprecise ("fuzzy") words from natural language. Fuzzy techniques have been invented with the purpose to translate these rules into a precise dependence $y = \widetilde{f}(x_1, \ldots, x_n)$. For deterministic dependencies $y = f(x_1, \ldots, x_n)$, there are universal approximation results according to which for each continuous function on a bounded domain and for every $\varepsilon > 0$, there exist fuzzy rules for which the resulting approximate dependence $\widetilde{f}(x_1, \ldots, x_n)$ is ε-close to the original function $f(x_1, \ldots, x_n)$.
In practice, many dependencies are *random*, in the sense that for each combination of the values $x_1, \ldots, x_n$, we may get different values y with different probabilities. It has been proven that fuzzy systems are universal approximators for such random dependencies as well. However, the existing proofs are very complicated and not intuitive. In this paper, we provide a simplified proof of this universal approximation property.

1 Formulation of the Problem

It is important to determine dependencies. One of the main objectives of science is to find the state of the world and to predict the future state of the world – both in situations when we do not interfere and when we perform a certain action. The state of the world is usually characterized by the values of appropriate physical quantities.

For example:

- we would like to know the distance y to a distant star,
- we would like to predict tomorrow's temperature y at a given location, etc.

In some cases, we can directly measure the current value of the quantity y of interest. However, in many practical cases, such a direct measurement is not possible – e.g.:

- while it is possible to measure a distance to a nearby town by just driving there,

– it is not yet possible to directly travel to a faraway star.

And it is definitely not possible to measure tomorrow's temperature y today.

In such situations, since we cannot directly measure the value of the desired quantity y, a natural idea is:

– to measure related easier-to-measure quantities $x_1, \ldots, x_n$, and then
– to use the known dependence $y = f(x_1, \ldots, x_n)$ between these quantities to estimate y.

For example, to predict tomorrow's temperature at a given location, we can:

– measure today's values of temperature, wind velocity, humidity, etc. in nearby locations, and then
– use the known equations of atmospheric physics to predict tomorrow's temperature y.

In some cases we know the exact form of the dependence $y = f(x_1, \ldots, x_n)$, but in many other practical situations, we do not have this information. Instead, we have to rely on experts who often formulate their rules in terms of imprecise ("fuzzy") words from natural language.

Imprecise ("fuzzy") rules and how they can be transformed into formulas. What kind of imprecise rules can we have? In some cases, the experts formulating the rule are imprecise both about x_i and about y. In such situations, we may have rules like this: "if today's temperature is very low and the Northern wind is strong, the temperature will remain very low tomorrow." In this case, x_1 is temperature today, x_2 is the speed of the Northern wind, y is tomorrow's temperature, and the properties "very low" and "strong" are imprecise.

In general, we have rules of the type

$$\text{"if } x_1 \text{ is } A_{k1}, \ldots, \text{ and } x_n \text{ is } A_{kn}, \text{ then } y \text{ is } A_k\text{"},$$

where A_{ki} and A_k are imprecise properties.

It is worth mentioning that in some cases, the information about x_i is imprecise, but the conclusion about y is described by a precise expression. For example, in non-linear mechanics, we can say that when the stress x_1 is small, the strain y is determined by a linear formula $y = k \cdot x_1$, with known k, but when the stress is high, we need to use a nonlinear expression $y = k \cdot x_1 - a \cdot x_2^2$ with known k and a. Here, both expressions are exactly known, but the condition when to apply one or another is described in terms of imprecise words like "small".

To transform such expert rules into a precise expression, Zadeh invented fuzzy logic; see, e.g., [1, 4, 5]. In fuzzy logic, to describe each imprecise property P, we ask the expert to assign, to each possible value x of the corresponding quantity, a degree $\mu_P(x)$ to which the value x satisfies this property – e.g., to what extent the value x is small. We can do this, e.g., by asking the expert to mark, on a scale from 0 to 10 to what extent the given value x is small. If the

expert marks 7, we take $\mu_P(x) = 7/10$. The function $\mu_P(x)$ that assigns this degree is known as the *membership function* corresponding to the property P.

For given inputs $x_1, \ldots, x_n$, a value y is possible if it fits within one of the rules, i.e., if:

- either the first rule is satisfied, i.e., x_1 is A_{11}, $\ldots$, x_n is A_{1n}, and y is A_1,
- or the second rule is satisfied, i.e., x_1 is A_{21}, $\ldots$, x_n is A_{2n}, and y is A_2, etc.

Since we assumed that we know the membership functions $\mu_{ki}(x_i)$ and $\mu_k(y)$ corresponding to the properties A_{ki} and A_k, we can thus find the degrees $\mu_{ki}(x_i)$ and $\mu_k(y)$ to which each corresponding property is satisfied.

To estimate the degree to which y is possible, we must be able to deal with propositional connectives "or" and "and", i.e., to come up with a way to estimate our degrees of confidence in statements $A \vee B$ and $A \,\&\, B$ based on the known degrees of confidence a and b of the elementary statements A and B. In fuzzy logic, such estimation algorithms are known as *t-conorms* ("or"-operations) and *t-norms* ("and"-operations). We will denote them by $f_\vee(a, b)$ and $f_\&(a, b)$. In these terms, the degree $\mu(y)$ to which each value y is possible can be estimated as $\mu(y) = f_\vee(r_1, r_2, \ldots)$, where

$$r_k \stackrel{\text{def}}{=} f_\&(\mu_{k1}(x_1), \ldots, \mu_{kn}(x_n), \mu_k(y)).$$

We can then transform these degrees into a numerical estimate $\overline{y}$. This can be done, e.g., by minimizing the weighted mean square difference $\int \mu(y) \cdot (y - \overline{y})^2 \, dy$, which results in

$$\overline{y} = \frac{\int y \cdot \mu(y) \, dy}{\int \mu(y) \, dy}.$$

Universal approximation result for deterministic dependencies. For deterministic dependencies $y = f(x_1, \ldots, x_n)$, there are universal approximation results according to which for each continuous function on a bounded domain and for every $\varepsilon > 0$, there exist fuzzy rules for which the resulting approximate dependence $\widetilde{f}(x_1, \ldots, x_n)$ is ε-close to the original function $f(x_1, \ldots, x_n)$ for all the values x_i from the given domain.

In practice, we can often only make probabilistic predictions. In practice, many dependencies are *random*, in the sense that for each combination of the values $x_1, \ldots, x_n$, we may get different values y with different probabilities.

Fuzzy systems are universal approximators for random dependencies as well. It has been proven that fuzzy systems and universal approximators for random dependencies as well; see, e.g., [2, 3].

Remaining problem: can we simplify these proofs. The proofs presented in [2, 3] are very complicated and not intuitive. It is therefore desirable to simplify these proofs.

What we do in this paper. In this paper, we provide a simplified proof of the universal approximation property for random dependencies.

2 Towards a Simplified Proof

Main idea: how do we simulate random dependencies? To simulate a deterministic dependence $y = f(x_1, \ldots, x_n)$, we design an algorithm that, given the values $x_1, \ldots, x_n$, computes the corresponding value y.

To simulate a random dependence, a computer must also use the results of some *random number generators* that generate numbers distributed according to some probability distribution. Such generators are usually based on the basic random number generator – which is either supported by the corresponding programming language or even on a hardware level – that generates numbers uniformly distributed on the interval $[0, 1]$.

From this viewpoint, the result of simulating a random dependency has the form

$$y = F(x_1, \ldots, x_n, \omega_1, \ldots, \omega_m),$$

where F is the corresponding algorithm, x_i are inputs, and the values ω_j comes from the basic random number generator.

In these terns, what does it mean to approximate? In the above terms, to approximate means to find a function $\widetilde{F}$ for which, for all possible inputs x_i from the given bonded range, and for all possible values ω_j, the corresponding value

$$\widetilde{y} = \widetilde{F}(x_1, \ldots, x_n, \omega_1, \ldots, \omega_m)$$

are ε-close to the results of applying the algorithm F to the same values x_i and ω_j.

This leads to a simplified proof. The above idea leads to following simplified proof:

- due to the universal approximation theorem for deterministic dependencies, for every $\varepsilon > 0$, there exists a system of fuzzy rules for which the value of the corresponding function $\widetilde{F}$ is ε-close to the value of the original function F;
- thus, we get a fuzzy system of rules that provides the desired approximation to the original random dependency F.

Acknowledgments

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On the representation of the d_p metric as a fuzzy set

Juan Carlos Figueroa-García

Universidad Distrital Francisco José de Caldas, Bogotá – Colombia
jcfigueroag@udistrital.edu.co *

Abstract. This paper compares different d_p metrics to represent them as fuzzy sets. The obtained evidence can help to represent human-like perceptions about distances. An application example is provided, and some interpretation issues are explained.

1 Introduction and Motivation

The d_p family of metrics is among the most used in engineering, mathematics, etc. and its properties are well known. Its application to fuzzy theory is wide as well, so the relationship between d_p metrics and human-like reasoning is worth been investigated.

The classical d_p family of metrics has been defined for crisp sets/elements, and some applications to fuzzy sets have been proposed by Xuechang [1], Chaudhuri & Rosenfeld [2], and Figueroa-García et al [3], [4], [5], [6]. Some additional properties of the d_p metrics over fuzzy numbers can help to represent how humans perceive distances between objects/sets. A natural thinking comes from the distance used to measure how close/far is an object from an observer, and how that distance affects the perception of closeness of the observer.

2 The classical d_p metric

A metric d is a function $d : X \times X \to \mathbb{R}$ that is non-negative, symmetric, subadditive (triangle inequality), and indiscernible ($d(x, y) = 0$ if and only if $x = y$). Let $X = \mathbb{R}^n$ and let $x, y \in X$, then the Minkowsky metric $d_p(x, y)$ is as follows:

$$d_p(x, y) := \left(\sum_i |x_i - y_i|^p \right)^{1/p}. \tag{1}$$

When $p = 1$ it is often called *Manhattan distance*, for $p = 2$ it is called *Euclidean distance*. We can define d_∞ as the limit of d_p when $p \to \infty$ which is

* Juan Carlos Figueroa-García is Assistant Professor of the Universidad Distrital Francisco José de Caldas, Bogotá – Colombia.

known as the *Chebyshev distance*:

$$\lim_{p\to\infty}\left(\sum_i |x_i - y_i|^p\right)^{1/p} = \max_i |x_i - y_i|. \tag{2}$$

The d_p metric is rarely used for $p < 1$ since it stops being a metric. However, $p < 0$ have been used as a reference for some IEEE standards in consumer electronics.

This way, we can easily observe that d_p is monotonically decreasing for $p \geqslant 1$:

$$d_1(x,y) \geqslant d_2(x,y) \geqslant d_3(x,y) \geqslant \cdots \geqslant d_\infty(x,y) = \max_i |x_i - y_i|. \tag{3}$$

3 The d_p metric for comparing fuzzy numbers

Let $\mathcal{P}(X)$ be the class of all crisp sets, and $\mathcal{F}(X)$ be the class of all fuzzy sets. A fuzzy set A is defined over a universe of discourse X and is characterized by a membership function $\mu_A(x)$ that takes values in the interval [0,1], $A: X \to [0,1]$. Thus, a fuzzy set A is a set of ordered pairs of an element x and its membership degree, $\mu_A(x)$, i.e.,

$$A = \{(x, \mu_A(x)) \,|\, x \in X\}. \tag{4}$$

Then, a fuzzy number (see Bede [7] and Diamond & Kloeden [8]) is defined as follows:

Definition 1 *Consider a fuzzy subset of the real line $A : \mathbb{R} \to [0,1]$. Then A is a **fuzzy number (FN)** if it satisfies the following properties:*

*i) A is **normal**, i.e. $\exists x' \in \mathbb{R}$ such that $A(x') = 1$;*
*ii) A is **α-convex** (i.e. $A(\alpha x + (1-\alpha)y) \geqslant \min\{A(x), A(y)\}$, $\forall\, \alpha \in [0,1]$);*
*iii) A is **upper semicontinuous** on $\mathbb{R}$, i.e. $\forall \epsilon > 0\, \exists \delta > 0$ such that*

$$A(x) - A(x') < \epsilon,\ |x - x'| < \delta;$$

*iii) A is **compactly supported** i.e. its closure $cl(A) = \{x : A(x) > 0\}$ is compact.*

Let us denote $\mathcal{F}_1(\mathbb{R})$ as the class of all FNs. The α-*cut* of a set $A \in \mathcal{F}_1(\mathbb{R})$ namely ${}^\alpha A$ is the set of values with a membership degree equal or greatest than α, this means:

$${}^\alpha A = \{x \,|\, \mu_A(x) \geqslant \alpha\}\ \forall\ x \in X, \tag{5}$$

$${}^\alpha A = \Big[\inf_x {}^\alpha\mu_A(x),\ \sup_x {}^\alpha\mu_A(x)\Big] = \big[\,\check{A}_\alpha,\ \hat{A}_\alpha\,\big]. \tag{6}$$

3.1 d_p distance between FNs

According to Ramík and Řimánek [9], two α-convex fuzzy sets A, B are equal if and only if ${}^{\alpha}A = {}^{\alpha}B \;\forall\; \alpha \in [0,1]$, ${}^{\alpha}A := [\inf {}^{\alpha}A, \sup {}^{\alpha}A]$ and ${}^{\alpha}B := [\inf {}^{\alpha}B, \sup {}^{\alpha}B]$. This leads us to think that the distance d between two α-convex fuzzy sets A, B is $d(A,B) = 0$ if and only if $d({}^{\alpha}A,\ {}^{\alpha}B) = 0 \,\forall\, \alpha \in [0,1]$ which means that $A = B$, otherwise there are differences between A, B. Now, the Minkowski distance between two FNs d_p can be defined as follows:

Definition 2 *Let $A, B \in \mathcal{F}_1(X)$ be two FNs. The distance (metric) d_p between A and B given a set of n α-cuts, $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $\Lambda = \sum_{i=1}^{n} \alpha_i$, is:*

$$d_p(A,B) := \frac{1}{\Lambda} \sum_{i=1}^{n} \alpha_i \left(\left| \check{A}_{\alpha_i} - \check{B}_{\alpha_i} \right|^p + \left| \hat{A}_{\alpha_i} - \hat{B}_{\alpha_i} \right|^p \right)^{1/p}. \tag{7}$$

For continuous α we have that $\Lambda = \int_0^1 \alpha \, d\alpha = 1/2$, so d_p can be defined as:

Definition 3 *Let $A, B \in \mathcal{F}_1(X)$ be two FNs. The distance (metric) d_p between A and B given continuous α and $\Lambda = 1/2$, is:*

$$d_p(A,B) := 2 \int_0^1 \alpha \left(\left| \check{A}_{\alpha} - \check{B}_{\alpha} \right|^p + \left| \hat{A}_{\alpha} - \hat{B}_{\alpha} \right|^p \right)^{1/p} d\alpha. \tag{8}$$

We can define d_∞ as the limit of d_p when $p \to \infty$:

$$d_\infty(A,B) := \frac{1}{\Lambda} \sum_{i=1}^{n} \alpha_i \left(\max \left(\left| \check{A}_{\alpha_i} - \check{B}_{\alpha_i} \right|, \left| \hat{A}_{\alpha_i} - \hat{B}_{\alpha_i} \right| \right) \right). \tag{9}$$

And for continuous α:

$$d_\infty(A,B) := 2 \int_0^1 \alpha \left(\max \left(\left| \check{A}_{\alpha} - \check{B}_{\alpha} \right|, \left| \hat{A}_{\alpha} - \hat{B}_{\alpha} \right| \right) \right). \tag{10}$$

By extension of Eq. (3), we can also see a similar behavior of $d_p(A,B)$:

$$d_1(A,B) \geqslant d_2(A,B) \geqslant d_3(A,B) \geqslant \cdots \geqslant d_\infty(A,B).$$

3.2 d_p distance and human-like reasoning

It is clear that p is a degree of freedom in both $d_p(x,y)$ and $d_p(A,B)$, so the perception of closeness of two objects depends on the observer. A distance can be perceived in different ways by different people, so p can help to represent the perception of the observer around $d_p(x,y)$ and $d_p(A,B)$.

This way, we propose to use p as a parameter of a fuzzy set that represents closeness between two sets A, B based on the distance $d_p(x,y)$ and $d_p(A,B)$. To

do so, a useful membership function to represent *closeness* could be:

$$\mu_c(d_p, q) = \frac{1}{1 + \left|\frac{d_p}{q}\right|^{2p}}, \tag{11}$$

$$\mu_c(d_p, q) = \frac{1}{1 + \left|\frac{d_p}{q}\right|^{2p}}, \tag{12}$$

where p is the order of the selected metric, and q is a shape parameter.

The parameter q can be interpreted as a tolerance of the observer around d_p, so as bigger q as wider μ_c. Another interesting fact is that for a given q, μ_c holds the following property:

$$\mu_c(q, q) = 0.5,$$

which means that μ_c is wide/tight around q in a nonlinear shape that depends on d_p.

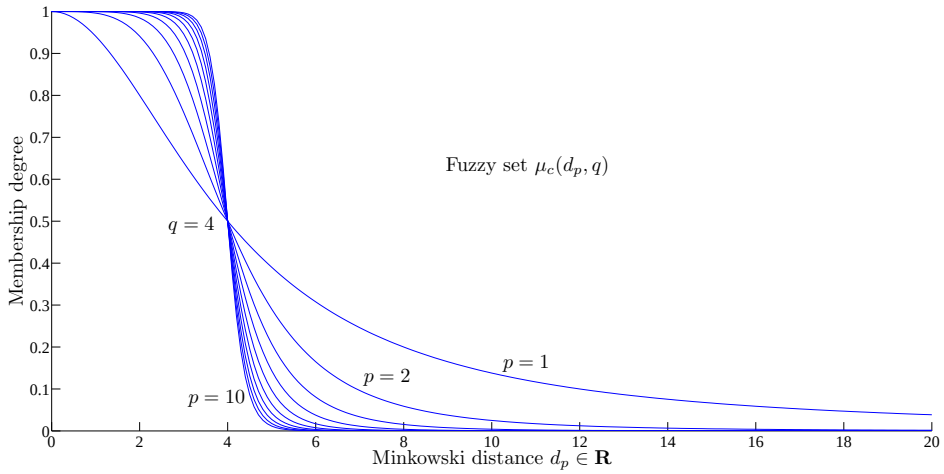

Fig. 1. Fuzzy set $\mu_c(d_p, q)$ for different values of p and $q = 4$

Figure 1 shows $\mu_c(d_p, q)$ for $p = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $q = 4$. Note that $\mu_c(d_p, 1)$ is much wider than $\mu_c(d_p, 10)$ which is much tighter.

This way, we can use p as a shape operator of $\mu_c(d_p, q)$ in order to represent the perception of an observer about specific values of d_p. Also p could be seen as the observer tolerance about the distance between tow objects/sets. This can help to design fuzzy logic systems and rule based systems based on distances between objects/sets.

Further discussion about this proposal is needed, and other shapes should be explored to represent how humans perceive distances and other measures like the Hausdorff metric, Mahalanobis metric, etc.

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Why Decimal System and Binary System Are the Most Widely Used: A Possible Explanation

Gerardo Muela

Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
gdmuela@miners.utep.edu

Abstract. What is so special about numbers 10 and 2 that decimal and binary systems are the most widely used? One interesting fact about 10 is that when we start with a unit interval and we want to construct an interval of half width, then this width is exactly 5/10; when we want to find a square of half area, its sides are almost exactly 7/10, and when we want to construct a cube of half volume its sides are almost exactly 8/10. In this paper, we show that 2, 4, and 10 are the only numbers with this property – at least among the first billion numbers. This may be a possible explanation of why decimal and binary systems are the most widely used.

1 Formulation of the Problem

Problem. What is so special about numbers 10 and 2 that decimal and binary systems are the most widely used?

This questions was raised, e.g., in [1].

Observation. One interesting fact about 10 is the following:

- When we start with a unit interval and we want to constrict an interval of half width, then this width is exactly $1/2 = 5/10$.
- When we start with a unit square and want to find a square of area $1/2$, its sides are $\sqrt{1/2}$, which is almost exactly $7/10$:

$$\left|\sqrt{\frac{1}{2}} - \frac{7}{10}\right| < \frac{1}{100}.$$

- When we start with a unit cube and want to find a cube of volume $1/2$, its sides are $\sqrt[3]{1/2}$, which is almost exactly $8/10$:

$$\left|\sqrt[3]{\frac{1}{2}} - \frac{8}{10}\right| < \frac{1}{100}.$$

So, whether we want to construct a piece of land which is (almost) exactly of half-area, or a piece of gold which is (almost) exactly of half-volume, decimal systems is very convenient.

Are there any other numbers with this property? Maybe here are other bases b with this property, i.e., bases b for which, for appropriate numbers n_1, n_2, and n_3, we have

$$\left|\frac{1}{2}-\frac{n_1}{b}\right|<\frac{1}{b^2},\quad \left|\sqrt{\frac{1}{2}}-\frac{n_2}{b}\right|<\frac{1}{b^2},\quad \left|\sqrt[3]{\frac{1}{2}}-\frac{n_3}{b}\right|<\frac{1}{b^2}. \tag{1}$$

What we do in this paper. In this paper, we show that – at least among the first billion numbers b – only the numbers $b=2$, $b=4$, and $b=10$ satisfy this property.

Base 4 is, in effect, the same as the binary system – we just group two binary digits to get one 4-ary digit, just like we get an 8-ary system when we group three binary digits or 16-based system when we group 4 binary digits together.

Thus, the above result may be a good explanation of why decimal and binary systems are the most widely used.

2 Analysis of the Problem

Considering the first condition. Let us first consider the first of the desired inequalities: $\left|\frac{1}{2}-\frac{n_1}{b}\right|<\frac{1}{b^2}$. When the base is even, i.e., when $b=2k$ for some integer k, then this property is clearly satisfied: indeed, in this case, for $n_1=k$, we get $\frac{n_1}{b}=\frac{1}{2}$ and thus, $\left|\frac{1}{2}-\frac{k}{b}\right|=0<\frac{1}{b^2}$.

On the other hand, if b is odd, i.e., if $b=2k+1$ for some natural number $k\ge 1$, then, for $\frac{1}{2}=\frac{k+0.5}{2k+1}=\frac{k+0.5}{b}$, the closest fractions of the type $\frac{n_1}{b}$ are the fractions $\frac{k}{b}$ and $\frac{k+1}{b}$. For both these fractions, we have

$$\left|\frac{k+0.5}{2k+1}-\frac{k}{2k+1}\right|=\left|\frac{k+0.5}{2k+1}-\frac{k+1}{2k+1}\right|=\frac{0.5}{2k+1}=\frac{1}{2\cdot(2k+1)}=\frac{1}{2b}.$$

The desired inequality thus takes the form $\frac{1}{2b}<\frac{1}{b^2}$, which is equivalent to $2b>b^2$ and $2>b$. However, odd bases start with $b=3$. So, the first condition cannot be satisfied by odd bases b.

Thus, the first condition is equivalent to requiring that the base b is an even number.

How do we check the second condition. If we check the second condition $\left|\sqrt{\frac{1}{2}}-\frac{n_2}{b}\right|<\frac{1}{b^2}$ literally, then we need to consider all possible values n_2 from 0

to b. However, this can avoided if we multiply both sides of the desired inequality by b and consider the equivalent inequality $\left|b \cdot \sqrt{\frac{1}{2}} - n_2\right| < \frac{1}{b}$. In this case, we can easily see that n_2 is the nearest integer to the product $b \cdot \sqrt{\frac{1}{2}}$:

$$n_2 = \left[b \cdot \sqrt{\frac{1}{2}}\right],$$

where $[x]$ denotes the nearest integer to the real number x. In these terms, the desired inequality takes the form

$$\left|b \cdot \sqrt{\frac{1}{2}} - \left[b \cdot \sqrt{\frac{1}{2}}\right]\right| < \frac{1}{b}. \tag{2}$$

This is the inequality that we will check.

How to check the third condition. Similarly, if we check the third condition $\left|\sqrt[3]{\frac{1}{2}} - \frac{n_3}{b}\right| < \frac{1}{b^2}$ literally, then we need to consider all possible values n_3 from 0 to b. However, this can avoided if we multiply both sides of the desired inequality by b and consider the equivalent inequality $\left|b \cdot \sqrt[3]{\frac{1}{2}} - n_3\right| < \frac{1}{b}$. In this case, we can easily see that n_3 is the nearest integer to the product $b \cdot \sqrt[3]{\frac{1}{2}}$:

$$n_3 = \left[b \cdot \sqrt[3]{\frac{1}{2}}\right],$$

where $[x]$ denotes the nearest integer to the real number x. In these terms, the desired inequality takes the form

$$\left|b \cdot \sqrt[3]{\frac{1}{2}} - \left[b \cdot \sqrt[3]{\frac{1}{2}}\right]\right| < \frac{1}{b}. \tag{3}$$

This is the inequality that we will check.

The checking. For each even number b from 2 to 10^9, we checked whether this number satisfies both conditions (2) and (3). A simple Java program for this checking is given in the appendix.

The result of the checking. The result is that among all the bases b from 1 to 10^9, both roots are only well approximated for $b = 2$, $b = 4$, and $b = 10$. Thus, only for these three bases, the desired condition (1) is satisfied.

This may explain why decimal and binary systems are the most frequently used.

Natural conjecture. We have checked all the values b until 10^9. This makes us conjecture that out of *all* possible natural numbers $b \geq 2$, only the numbers 2, 4, 10 satisfy the property (1).

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A Code

```
public static void main(String [] args){
  double value;
  //Loop that iterates from 2 to 10^9
  for(int b = 2; b <= 1000000000; b += 2){
    value = Math.sqrt(0.5) * b;
    //Checks if the square root is well approximated
    if(Math.abs(value - Math.round(value)) < 1. / b){
      value = Math.cbrt(0.5) * b;
      //Checks if the cubic root is well approximated
      if(Math.abs(value - Math.round(value)) < 1. / b){
        System.out.println("Square and cubic roots "
          + "are well approximated in base " + b);
      }
    }
  }
}
```

Attraction-Repulsion Forces Between Biological Cells: A Theoretical Explanation of Empirical Formulas

Olga Kosheleva, Martine Ceberio, and Vladik Kreinovich

University of Texas at El Paso, El Paso, TX 79968, USA
olgak@utep.edu, mceberio@utep.edu, vladik@utep.edu

Abstract. Biological calls attract and repulse each other: if they get too close to each other, they repulse, and if they get too far away from each other, they attract. There are empirical formulas that describe the dependence of the corresponding forces on the distance between the cells. In this paper, we provide a theoretical explanation for these empirical formulas.

1 Formulation of the Problem

Biological calls interact. Biological cells attract and repulse each other. For each type of cell, there is a certain distance R_1 at which there is no interaction.

- When the cells get closer to each other than this threshold distance, i.e., when the distance r between the cells becomes smaller than R_1, then the cells repulse each other.
- On the other hand, if the two cells deviate further away that the threshold distance R_1, they start attracting each other.

As a result of these two forces, the cells stay at the same – biologically optimal – distance from each other.

Empirical formulas describing interaction between the cells. According to [2, 3, 5], the interaction force $\mathbf{f}$ between the two cells at a distance $\mathbf{r}$ has the following form:

- when $r < R_1$, we have $\mathbf{f} = -k_1 \cdot \left(\frac{1}{r} - \frac{1}{R_1}\right) \cdot \mathbf{e}$, where r is the length of the vector $\mathbf{r}$ (i.e., the distance between the cells) and $\mathbf{e} \stackrel{\text{def}}{=} \frac{\mathbf{r}}{r}$ is the unit vector in the direction $\mathbf{r}$;
- when $r > R_1$, we have $\mathbf{f} = k_2 \cdot (r - R_1) \cdot \mathbf{e}$.

Formulation of the problem. How can we explain these empirical formulas?

What we do in this paper. In this paper, we provide a theoretical explanation for the above empirical formulas.

2 Analysis of the Problem

Qualitative requirements. We want to find the dependence $f(r)$ of the interactive force f on the distance r between the two cells.

To find such a dependence, let us consider natural requirements on $f(r)$.

Monotonicity. The larger the difference between the actual distance r and the desired distance R_1, the larger should be the force. Thus:

- the repulsion force should increase when the distance r decreases, while
- the attraction force should increase as the distance r increases.

It should be mentioned that the empirical formulas satisfy this property – i.e., the corresponding dependencies $f(r)$ are monotonic.

Analyticity. This is a general phenomenon in physics, that all dependencies in fundamental physics are described by analytical functions, i.e., by functions which can be expanded in Taylor or, more generally, by Laurent series; see, e.g., [4]. For functions of one variable, this means that we must have

$$f(r) = a_0 + a_1 \cdot r + a_2 \cdot r^2 + \ldots + a_{-1} \cdot r^{-1} + a_{-2} \cdot r^{-2} + \ldots$$

In fundamental physics, this phenomenon is usually explained by the need to consider quantum effects: quantum analysis means extension to complex numbers – and analytical functions are, in effect, differential functions of complex variables; see, e.g., [4].

Again, it is worth mentioning that both empirical formulas – the formula corresponding to $r < R_1$ and the formula corresponding to $r > R_1$ – are analytical.

Tidal forces. The main objective of the forces between the two cells are to keep the cells at a certain distance. This motivates the direct effect of the forces:

- when the cells are two close, the repulsion force will make move apart, while
- when the cells are too far away from each other, the attraction force will make them get closer.

However, with this direct effect, there is also an undesired side effect, caused by the fact that cells are not points. As a result, different parts of the cell have slightly difference force acting on them. So, in addition to the overall force that makes the cell move in the desired direction, we also have tidal forces that make the parts of the cell move with respects to each other – i.e., make the cell compress or stretch.

In general, the tidal forces are proportional to the gradient of the force field (see, e.g., [4]), i.e., in this case, to the derivative $F(r) \stackrel{\text{def}}{=} \dfrac{df}{dr}$.

From the biological viewpoint, tidal forces are undesirable, so they should be as small as possible.

Scale invariance. Physical laws are formulated in terms of the numerical values of physical quantities. However, these numerical values depend on what measuring unit we select to describe this quantity.

For example, if we first measure distances in meters, and then start using centimeters instead, then all the numerical values get multiplied by a factor of 100. In particular, 2 m becomes 200 cm.

In most fundamental physical laws, there is no physically preferred unit. In such situations, it make sense to require that the physical law not depend on the choice of the unit.

Of course it does not means that all the formulas remain unchanged if we simply change the measuring unit of one of the quantities. Usually, if we change the unit of one of the quantities, then we have to accordingly change the units of related quantities. However, after an appropriate re-scaling of all the units, all the formulas should remain the same.

In precise terms, scale-invariance of the dependence $b = B(a)$ between two quantities a and b means that for every λ, there exists a $\mu(\lambda)$ such that if we change a to $a' = \lambda \cdot a$ and b to $b' = \mu(\lambda) \cdot b$, the dependence remain the same: if $b = B(a)$, then we should have $b' = B(a')$, i.e., $\mu(\lambda) \cdot b = B(\lambda \cdot a)$.

For the dependence of the force itself of the distance, there is clearly no scale-invariance: indeed, in this case, there is a special distance R_1 at which the force is 0. However, for the *tidal force* $F(r)$, interestingly, there is scale-invariance: namely, $F(r) \sim r^{-2}$ for small r and $F(r) = \text{const}$ for large r; both are scale-invariant formulas.

Now, we are ready to describe our result.

3 Definitions and the Main Result

The above properties take the following form:

Definition 1.

- *By a* force function*, we mean a function $f(r)$ from positive numbers to real numbers.*
- *We say that a force function unction $f(r)$ is* analyticaL *if it can be expanded in Laurent series*

$$f(r) = a_0 + a_1 \cdot r + a_2 \cdot r^2 + \ldots + a_{-1} \cdot r^{-1} + a_{-2} \cdot r^{-2} + \ldots$$

- *We say that a force function is* monotonic-at-0 *if for sufficiently small r, its absolute value increases as r decreases.*
- *We say that a force function is* monotonic-at-∞ *if for sufficiently large r, its absolute value increases as r increases.*
- *By a* tidal force function *corresponding to the force function $f(r)$, we mean its derivative $F(r) = \dfrac{df}{dr}$.*
- *We say that a tidal force function is* scale-invariant *if for every $\lambda > 0$, there exists a $\mu(\lambda)$ for which, for all r and a, $a = F(r)$ implies that $\mu(\lambda) \cdot a = F(\lambda \cdot r)$.*

Next, we should describe the property that the tidal forces should not grow too fast.

Definition 2. *Let $f(r)$ be an analytical monotonic-at-0 force function $f(r)$, let $F(r)$ be its tidal force function, and let $F(r)$ be scale-invariant. We say that $F(r)$* grows fast *if there exists another analytical monotonic-at-0 force function $g(r)$, with scale-invariant tidal force function $G(r)$, for which $\dfrac{F(r)}{G(r)} \to \infty$ as $r \to 0$.*

Definition 3. *Let $f(r)$ be an analytical monotonic-at-∞ force function $f(r)$, let $F(r)$ be its tidal force function, and let $F(r)$ be scale-invariant. We say that $F(r)$* grows fast *if there exists another analytical monotonic-at-∞ force function $g(r)$, with scale-invariant tidal force function $G(r)$, for which $\dfrac{F(r)}{G(r)} \to \infty$ as $r \to 0$.*

Proposition 1. *Every analytical monotonic-at-0 force function $f(r)$ for which the tidal force function $F(r)$ is scale-invariant and does not grow fast, has the form $f(r) = \dfrac{c_0}{r} + c_1$ for some c_0 and c_1.*

Proposition 2. *Every analytical monotonic-at-∞ force function $f(r)$ for which the tidal force function $F(r)$ is scale-invariant and does not grow fast, has the form $f(r) = c_0 \cdot r + c_1$ for some c_0 and c_1.*

Discussion. These are exactly the empirical formulas that we wanted to explain. Thus, we have a theoretical explanation for these formulas.

4 Proofs

1°. Let us first see what we can conclude from scale-invariance of the tidal force function $F(r)$. By definition, this scale-invariance means that

$$F(\lambda \cdot r) = \mu(\lambda) \cdot F(r).$$

The function $F(r)$ is analytical, thus smooth. For smooth functions, every function $F(r)$ with this property has the form $F(r) = c \cdot r^\alpha$ for some constants c and α; see, e.g., [1].

This fact is easy to prove. Since the function $F(r)$ is smooth, the function $\mu(\lambda)$, which is equal to the ratio of two smooth functions $\mu(\lambda) = \dfrac{F(\lambda \cdot r)}{F(r)}$, is also smooth. Differentiating both sides of the equality $F(\lambda \cdot r) = \mu(\lambda) \cdot F(r)$ by λ and taking $\lambda = 1$, we conclude that $r \cdot \dfrac{dF}{dr} = \alpha \cdot F$, where $\alpha \stackrel{\text{def}}{=} \dfrac{d\mu}{d\lambda}_{|\lambda=1}$.

By moving all the terms containing r to one side and all the terms containing F to another side, we conclude that $\dfrac{dF}{F} = \alpha \cdot \dfrac{dr}{r}$. Integrating both sides, we get $\ln(F(r)) = \alpha \cdot \ln(r) + C$, for some integration constant C. Thus, for $F(r) = \exp(\ln(F(r)))$, we get $F(r) = c \cdot r^\alpha$, where $c \stackrel{\text{def}}{=} \exp(C)$.

2°. Since the force function $f(r)$ is analytical, its derivative is also analytical. Thus, α should be an integer.

For $\alpha = -1$, integration of the above expression for $F(r)$ would lead to $f(r) = c \cdot \ln(r)$, which is not an analytical function. Thus, $\alpha \neq -1$, and integration of $F(r)$ leads to $f(r) = c_0 \cdot r^{\alpha+1} + c_1$, where $c_0 \stackrel{\text{def}}{=} \dfrac{c}{\alpha+1}$.

3°. Monotonicity-at-0 implies that $\alpha + 1 < 0$, i.e., that that $\alpha + 1 \leq -1$ and $\alpha \leq -2$. For $\alpha < -2$, we could take $g(r) = r^{-1}$ with $G(r) = -r^{-2}$ and thus, $\dfrac{F(r)}{G(r)} \sim \dfrac{r^\alpha}{r^{-2}} = r^{\alpha+2}$. From $\alpha < -2$, it follows that $\alpha + 2 < 0$, hence $\dfrac{F(r)}{G(r)} \sim r^{\alpha+2} \to \infty$ as $r \to 0$. So, all the cases when $\alpha < -2$ correspond to the tidal force function that grows fast. The only case when this function does not grow fast is the case $\alpha = -2$, which leads to $f(r) = c_0 \cdot r^{-1} + c_1$.

4°. Similarly, monotonicity-at-∞ implies that $\alpha + 1 > 0$, i.e., that that $\alpha + 1 \geq 1$ and $\alpha \geq 0$.

For $\alpha > 0$, we could take $g(r) = r$ with $G(r) = 1$ and thus, $\dfrac{F(r)}{G(r)} \sim r^\alpha$. From $\alpha > 0$, $r^\alpha \to \infty$ as $r \to \infty$. So, all the cases when $\alpha > 0$ correspond to the tidal force function that grows fast. The only case when this function does not grow fast is the case $\alpha = 0$, which leads to $f(r) = c_0 \cdot r + c_1$.

The Propositions are proven.

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Why Unexpectedly Positive Experiences Make Decision Makers More Optimistic: An Explanation

Andrzej Pownuk and Vladik Kreinovich

Computational Science Program, University of Texas at El Paso
El Paso, TX 79968, USA, ampownuk@utep.edu, vladik@utep.edu

Abstract. Experiments show that unexpectedly positive experiences make decision makers more optimistic. However, there seems to be no convincing explanation for this experimental fact. In this paper, we show that this experimental phenomenon can be naturally explained within the traditional utility-based decision theory.

1 Formulation of the Problem

Experimental phenomenon. Experiments show that unexpectedly positive experiences make decision makers more optimistic. This was first observed on an experiment with rats [10]: rats like being ticked, and tickled rats became more optimistic. Several later papers showed that the same phenomenon holds for other decision making situations as well; see, e.g., [2, 7].

Similarly, decision makers who had an unexpectedly negative experiences became more pessimistic; see, e.g., [8].

Why: a problem. There seems to be no convincing explanation for this experimental fact.

What we do in this paper. In this paper, we show that this experimental phenomenon can be naturally explained within the traditional utility-based decision theory.

2 Formulating the Problem in Precise Terms

In precise terms, what does it mean to becomes more optimistic or less optimistic? The traditional utility-based decision theory describes the behavior of a rational decision maker in situations in which we know the probabilities of all possible consequences of each action; see, e.g., [1, 5, 6, 9]. This theory shows that under this rationality assumption, preferences of a decision maker can be described by a special function $U(x)$ called *utility function*, so that a rational decision maker selects an alternative a that maximizes the expected value $u(a)$ of the utility.

In this case, there is no such thing as optimism or pessimism: we just select the alternative which we know is the best for us.

The original theory describes the behavior of decision makers in situations in which we know the probability of each possible consequence of each action. In practice, we often have only *partial* information about these probabilities – and sometimes, no information at all. In such situations, there are several possible probability distributions consistent with our knowledge. For different distributions, we have, in general, different values of the expected utility. As a result, for each alternative, instead of the exact value of the expected utility, we have an *interval* $[\underline{u}(a), \overline{u}(a)]$ of possible values of $u(a)$. How can we make a decision based on such intervals?

In this case, natural rationality ideas lead to the conclusion that a decision should select an alternative a for which, for some real numbers $\alpha \in [0, 1]$, the combination $u(a) = \alpha \cdot \overline{u}(a) + (1 - \alpha) \cdot \underline{u}(a)$ is the largest possible; see, e.g., [4]. This idea was first proposed by the Nobelist Leo Hurwicz in [3].

The selection of α, however, depends on the person. The value $\alpha = 1$ means that the decision maker only takes into account the best possible consequences, and completely ignores possible less favorable situations. In other words, the values $\alpha = 1$ corresponds to complete optimism.

Similarly, the value $\alpha = 0$ means that the decision maker only takes into account the worst possible consequences, and completely ignores possible more favorable situations. In other words, the values $\alpha = 0$ corresponds to complete pessimism.

Intermediate values α mean that we take into account both positive and negative possibilities. The larger α, the close this decision maker to complete optimism. The smaller α, the closer the decision maker to complete pessimism. Because of this, the parameter α – known as the *optimism-pessimism index* – is a numerical measure of the decision maker's optimism.

In these terms:

- becoming more optimistic means that the value α increases, and
- becoming less optimistic means that the value α increases.

Thus, the above experimental fact takes the following precise meaning:

- if a decision maker has unexpectedly positive experiences, then this decision maker's α increases;
- if a decision maker has unexpectedly negative experiences, then this decision maker's α decreases.

This is the phenomenon that we need to explain.

3 Towards the Desired Explanation

Optimism-pessimism parameter α can be naturally interpreted as the subjective probability of positive outcome. The value α means that the

decision maker selects an alternative a for which the value $\alpha \cdot \overline{u}(a) + (1-\alpha) \cdot \underline{u}(a)$ is the largest possible.

Here, the value $\overline{u}(a)$ corresponds to the positive outcome, and the value $\underline{u}(a)$ corresponds to negative outcome.

For simplicity, let us consider the situation when we have only two possible outcomes:

- the positive outcome, with utility $\overline{u}(a)$, and
- the negative outcome, with utility $\underline{u}(a)$.

A traditional approach to decision making, as we have mentioned, assumes that we know the probabilities of different outcomes. In this case of uncertainty, we do not know the actual (objective) probabilities, but we can always come up with estimated (subjective) ones.

Let us denote the subjective probability of the positive outcome by p_+. Then, the subjective probability of the negative outcome is equal to $1 - p_+$. The expected utility is equal to $p_+ \cdot \overline{u}(a) + (1 - p_+) \cdot \underline{u}(a)$.

This is exactly what we optimize when we use Hurwicz's approach, with $\alpha = p_+$. Thus, the value α can be interpreted as the subjective probability of the positive outcome.

A new reformulation of our problem. In these terms, the above experimental phenomenon takes the following form:

- unexpectedly positive experiences increase the subjective probability of a positive outcome, while
- unexpectedly negative experiences decrease the subjective probability of a positive outcome.

To explain this phenomenon, let us recall where subjective probabilities come from.

Where subjective probabilities come from? A natural way to estimate the probability of an event is to consider all situations in which this event could potentially happen, and then take the frequency of this event – i.e., the ratio n/N of the number of times n when it happens to the overall number N of cases – as the desired estimate for the subjective probability. For example, if we flip a coin 10 times and it fell heads 6 times out of 10, we estimate the probability of the coin falling heads as 6/10.

Let us show that this leads to the desired explanation.

Resulting explanation. Suppose that a decision maker had n positive experiences in the past N situations. Then, the decision maker's subjective probability of a positive outcome is $p_+ = n/N$.

Unexpectedly positive experiences means that we have a series of new experiments, in which the fraction of positive outcomes was higher than the expected frequency p_+. In other words, unexpectedly positive experiences means that $n'/N' > p$, where N' is the overall number of new experiences, and n' is

the number of those new experiences in which the outcome turned out to be positive.

How will these new experiences change the decision maker's subjective probability? Now, the decision maker has encountered overall $N + N'$ situations, of which $n + n'$ were positive. Thus, the new subjective probability p'_+ is equal to the new ratio $p'_+ = \dfrac{n + n'}{N + N'}$. Here, by definition of p_+, we have

$$n = p_+ \cdot N$$

and, due to unexpected positiveness of new experiences, we have $n' > p_+ \cdot N'$. By adding this inequality and the previous equality, we conclude that $n + n' > p_+ \cdot (N + N')$, i.e., that

$$p'_+ = \frac{n + n'}{N + N'} > p_+.$$

In other words, unexpectedly positive experiences increase the subjective probability of a positive outcome.

As we have mentioned, the subjective probability of the positive outcome is exactly the optimism-pessimism coefficient α. Thus:

- the original subjective probability p_+ is equal to the original optimism-pessimism coefficient α, and
- the new subjective probability p'_+ is equal to the new optimism-pessimism coefficient α'.

So, the inequality $p'_+ > p_+$ means that $\alpha' > \alpha$, i.e., that unexpectedly positive experiences make the decision maker more optimistic. This is exactly what we wanted to explain.

Similarly, if we had unexpectedly negative experiences, i.e., if we had $n' < p_+ \cdot N'$, then we similarly get $n + n' < p_+ \cdot (N + N')$ and thus,

$$p'_+ = \frac{n + n'}{N + N'} < p_+$$

and $\alpha' < \alpha$. So, we conclude that unexpectedly negative experiences make the decision maker less optimistic. This is also exactly what we observe. So, we have the desired explanation.

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Why Convex Optimization Is Ubiquitous and Why Pessimism Is Widely Spread

Angel F. Garcia Contreras, Martine Ceberio, and Vladik Kreinovich

Department of Computer Science, University of Texas at El Paso
El Paso, TX 79968, USA
afgarciacontreras@miners.utep.edu,
mceberio@utep.edu, vladik@utep.edu

Abstract. In many practical applications, the objective function is convex. The use of convex objective functions makes optimization easier, but ubiquity of such objective function is a mystery: many practical optimization problems are not easy to solve, so it is not clear why the objective function – whose main goal is to describe our needs – would always describe easier-to-achieve goals. In this paper, we explain this ubiquity based on the fundamental ideas about human decision making. This explanation also helps us explain why in decision making under uncertainty, people often make pessimistic decisions, i.e.., decisions based more on the worst-case scenarios.

1 Why Convex Optimization Is Ubiquitous

Reasonable decision making means optimization. In many real life situations, we need to make a decision, i.e., we need to select an alternative x out of many possible alternatives.

Decision making theory has shown that the decision making of a rational person is equivalent to maximizing a special function $u(x)$ – known as *utility* – that describes this person's preferences; see, e.g., [1, 5, 6, 8]. Thus, maximization problems are very important for practical applications.

In many cases, the utility value is described by its monetary equivalent amount.

Small changes in an alternative should lead to small change in preferences, so the function $u(x)$ is usually continuous.

What if an optimization problem has several solutions? From the purely mathematical viewpoint, it is possible that an optimization problem has several solutions, i.e., several different alternatives $x^{(1)}$, $x^{(2)}$, ... all maximize the objective function $u(x)$:

$$u(x^{(1)}) = u(x^{(2)}) = \ldots = \max_x u(x).$$

From the practical viewpoint, however, the fact that, by using some criterion, we get several possible solutions, means that we can use this non-uniqueness to

optimize something else. For example, if a company selects a design x for a new plant, and several designs $x^{(1)}$, $x^{(2)}$, ... are equally profitable, then a reasonable idea is to select, among these most-profitable solutions, the one which is, e.g., the most environmentally friendly. This will weed out some of the possible designs. If even after taking into account environmental impact, we still have several possible alternatives, we can use the remaining non-uniqueness to optimize something else – e.g., look for the most aesthetically pleasing design. This process continues until we end up with the single optimal alternative.

In other words, if the objective function $u(x)$ allows several optimal solutions, this means, from the practical viewpoint, that we need to modify our preferences – i.e., in effect, modify the corresponding objective function – until we end up with an objective function that attains its maximum at the unique point.

So, while, from the mathematical viewpoint, we can consider arbitrary objective functions $u(x)$ – and they can serve as good approximations to the way we make decisions – the *final* objective function, the function that describes exactly how we actually make decisions, should have the unique maximum.

How can we describe such final objective functions? In general, selecting a decision x involves selecting the values of many different parameters $x_1, \ldots, x_n$ that characterize this decision. For example, when we select a design of a plant, we must take into account the land area that we need to purchase, the amount of steel and concrete that goes into construction, the overall length of roads, pipes, etc. forming the supporting infrastructure, etc.

Our original decision x is based on known costs of all these attributes. However, costs can change. If the cost per unit of the i-th attribute changes by the value d_i, then the overall cost of an option x changes from the original value $u(x)$ to the new value

$$u'(x) = u(x) + \sum_{i=1}^{n} d_i \cdot x_i. \tag{1}$$

It is therefore reasonable to select an objective function $u(x)$ in such away that not only the original function $u(x)$ has the unique maximum, but that for all possible combinations of values d_i, the resulting combination (1) also has the unique maximum.

Need to consider constraints. From the purely mathematical viewpoint, we often consider *unconstrained* optimization, where we have no prior restrictions on the values of the parameters $x_1, \ldots, x_n$ that describe the desired solution $x = (x_1, \ldots, x_n)$. In practice, there are always physical and economical restrictions on the possible values of these parameters. As a result, in practice, for each parameter x_i, we always have bounds $\underline{x}_i$ and $\overline{x}_i$, and we only consider values x_i from the corresponding intervals $[\underline{x}_i, \overline{x}_i]$.

Once we take into account the existence of constraints, we can always guarantee that the corresponding optimization problem always has a solution: indeed, on a bounded closed set $B = [\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$, every continuous function attains its maximum at some point $x \in B$.

Thus, we arrive at the following definition.

Definition 1. *A continuous function $u(x) = u(x_1, \ldots, x_n)$ is called a* final objective function *if for every combination of tuples $d = (d_1, \ldots, d_n)$, $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n)$, and $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)$ the following constrained optimization problem has the unique solution:*

$$\textit{Maximize } u(x) + \sum_{i=1}^{n} d_i \cdot x_i \textit{ under constraints } \underline{x}_i \le x_i \le \overline{x}_i.$$

Discussion. There is a class of functions which are realistic objective functions in the sense of the above definition – namely, the class of *strictly convex* functions $u(x)$, i.e., functions for which $u\left(\frac{x+x'}{2}\right) > \frac{u(x)+u(x')}{2}$ for all $x \ne x'$; see, e.g., [9]. Indeed, it is easy to prove that for a strictly convex function, maximum is attained at a unique point: if we have two different points $x \ne x'$ at which $u(x) = u(x') = \max\limits_{x} u(x)$, then, due to strong convexity, for the midpoint $x'' \stackrel{\text{def}}{=} \frac{x+x'}{2}$, we would have $u(x'') > u(x) = u(x')$, i.e., we would have $u(x'') > \max\limits_{x} u(x)$, which is not possible.

One can also easily check that if a function $u(x)$ is strictly convex, and if we add a linear expression $\sum\limits_{i=1}^{n} d_i \cdot x_i$ to this function, then the resulting sum $u'(x)$ is also strictly convex. Thus, strictly convex functions are indeed final objective functions in the sense of Definition 1.

Interestingly, if we restrict ourselves to smooth (at least three times differentiable) functions, the opposite is also true: only convex objective functions are final in the sense of the above definition.

Proposition 1. *Every smooth final objective function $u(x)$ is convex.*

Comments.

- This result explains why convex objective functions are ubiquitous in practical applications; see, e.g. [9].
- This result is also good for practical applications since, while optimization in general is NP-hard, feasible algorithms are known for solving convex optimization problem; see, e.g., [4, 7].

Proof of Proposition 1. Let us prove this by contradiction. Let us assume that there exists a smooth final objective function $u(x)$ which is not convex. A smooth function is convex if and only if at all points, its matrix of second derivatives is non-positive definite [9]. Since $u(x)$ is not convex, there exists a point p at which this matrix is not non-negative definite. At this point, the Taylor expansion of the function $u(x)$ has the form

$$u(x) = u(p) + \sum_{i=1}^{n} u_{,i} \cdot (x_i - p_i) + \frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} u_{,ij} \cdot (x_i - p_i) \cdot (x_j - p_j) + o((x-p)^2),$$

where $u_{,i} \stackrel{\text{def}}{=} \dfrac{\partial u}{\partial x_i}$ and $u_{,ij} \stackrel{\text{def}}{=} \dfrac{\partial^2 u}{\partial x_i \partial x_j}$. Thus, the function $u'(x) = u(x) - \sum\limits_{i=1}^{n} u_{,i} \cdot x_i$ has the form $u'(x) = q(x) + o((x-p)^2)$, where

$$q(x) \stackrel{\text{def}}{=} u'(p) + \frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} u_{,ij} \cdot (x_i - p_i) \cdot (x_j - p_j).$$

Let us take $\underline{x}_i = x_i^{(0)} - \varepsilon$ and $\overline{x}_i = x_i^{(0)} + \varepsilon$ for some small $\varepsilon > 0$. Then, for small $\varepsilon > 0$, $u(x)$ is very close to $q(x)$.

Non-negative definite would mean that $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} u_{,ij} \cdot (x_i - p_i) \cdot (x_j - p_j) \le 0$ for all x_i. The fact that the matrix $u_{,ij}$ is not non-negative definite means that there exists a vector $x_i - p_i$ for which $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} u_{,ij} \cdot (x_i - p_i) \cdot (x_j - p_j) > 0$. So, for a vector proportional to $x_i - p_i$ and which is within the box B, we have $q(x) > q(p)$. Thus, the maximum of the function $q(x)$ on the box B is *not* attained at p. Since the function $q(x)$ does not change if we reverse the sign of all the differences $x_i - p_i$, with each point $x = p + (x - p)$, the same maximum is attained at a different point $p - (x - p)$. So, for the function $q(x)$, the maximum is attained in at least two different points.

Let us now consider the original function $u'(x)$. If its maximum is attained at two different points, we get our contradiction. Let us now assume that its maximum m is attained at a single point y. This maximum is close to a maximum of the function $q(x)$. The fact that this function has only one maximum means that the value of $u'(x)$ at the point $p - (y - p)$ is slightly smaller than the value $m = u'(y)$. We can then take the plane (linear function) $u = m$, and, keeping its value to be m at the point y, we slightly rotate it and lower it until we touch some other point on the graph – close to $p - (y - p)$. This is possible for $q(x)$, thus it is possible for any function which is sufficiently close to $q(x)$ – in particular, for a function $u'(x)$ corresponding to a sufficiently small value $\varepsilon > 0$. Thus, we get a sum $u''(x)$ of $u'(x)$ and a linear function that has at least two maxima. Since $u'(x)$ is itself a sum of $u(x)$ and a linear function, this means that $u''(x)$ is also a sum of $u(x)$ and a linear function – so we get a contradiction with our assumption that the function $u(x)$ is a final objective function.

The proposition is proven.

2 Why Pessimism Is Widely Spread

Decision making under uncertainty. In many practical situations, we do not know the exact consequences of different actions. In other words, for each alternative x, instead of a single value $u(x)$, we have several different values $u(x, s)$ depending on the situation s. According to decision theory, in such situation, a reasonable idea is to optimize the so-called Hurwicz criterion

$$U(x) = \alpha \cdot \max_s u(x, s) + (1 - \alpha) \cdot \min_s u(x, s)$$

for some $\alpha \in [0,1]$; see, e.g., [2, 3, 5]. Here, $\alpha = 1$ corresponds to the optimistic approach, when we only consider the best-case scenarios, $\alpha = 0$ is pessimistic approach, when we only consider the worst cases, and $\alpha \in (0,1)$ means that we consider both the best and the worst cases.

When is this convex? From the viewpoint described in the previous section, it is reasonable to consider situations in which $u(x,s)$ is convex for every s *and* the objective function $U(x)$ is also convex.

For $\alpha = 0$, it is easy to show that the minimum of convex function is always convex; see, e.g., [9]. For $\alpha = 0.5$, we get the arithmetic average which is also convex. For $\alpha < 0.5$, we get a convex combination of cases $\alpha = 0$ and $\alpha = 0.5$, so we also get a convex functions.

However, for any $\alpha > 0.5$, this is no longer true. For example, let us take $s \in \{-,+\}$, with

$$u(x,+) = |x-1| \text{ and } u(x,-) = |x+1|.$$

Then, for every $\alpha > 0.5$, the function $U(x)$ attains its maximum value α at two difference points. Thus, $U(x)$ is not convex.

This explains why pessimism is widely spread. The fact that only in the pessimistic approach we can guaranteed that the resulting objective function is final explains why the pessimistic approach ($\alpha \le 0.5$) is widely spread.

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Which Value $\widetilde{x}$ Best Represents a Sample $x_1, \ldots, x_n$: Utility-Based Approach Under Interval Uncertainty

Andrzej Pownuk and Vladik Kreinovich

Computational Science Program
University of Texas at El Paso
El Paso, TX 79968, USA
ampownuk@utep.edu, vladik@utep.edu

Abstract. In many practical situations, we have several estimates $x_1, \ldots, x_n$ of the same quantity x. In such situations, it is desirable to combine this information into a single estimate $\widetilde{x}$. Often, the estimates x_i come with interval uncertainty, i.e., instead of the exact values x_i, we only know the intervals $[\underline{x}_i, \overline{x}_i]$ containing these values. In this paper, we formalize the problem of finding the combined estimate $\widetilde{x}$ as the problem of maximizing the corresponding utility, and we provide an efficient (quadratic-time) algorithm for computing the resulting estimate.

1 Which Value $\widetilde{x}$ Best Represents a Sample $x_1, \ldots, x_n$: Case of Exact Estimates

Need to combine several estimates. In many practical situations, we have several estimates $x_1, \ldots, x_n$ of the same quantity x. In such situations, it is often desirable to combine this information into a single estimate $\widetilde{x}$; see, e.g., [6].

Probabilistic case. If we know the probability distribution of the corresponding estimation errors $x_i - x$, then we can use known statistical techniques to find $\widetilde{x}$, e.g., we can use the Maximum Likelihood Method; see, e.g., [8].

Need to go beyond the probabilistic case. In many cases, however, we do not have any information about the corresponding probability distribution [6]. How can we then find $\widetilde{x}$?

Utility-based approach. According to the general decision theory, decisions of a rational person are equivalent to maximizing his/her *utility value* u; see, e.g., [1, 4, 5, 7]. Let us thus find the estimate $\widetilde{x}$ for which the utility $u(\widetilde{x})$ is the largest.

Our objective is to use a single value $\widetilde{x}$ instead of all n values x_i. For each i, the disutility $d = -u$ comes from the fact that if the actual estimate is x_i and we use a different value $\widetilde{x} \neq x_i$ instead, we are not doing an optimal thing. For example, if the optimal speed at which the car needs the least amount of fuel is x_i, and we instead run it at a speed $\widetilde{x} \neq x_i$, we thus waste some fuel.

For each i, the disutility d comes from the fact that the difference $\widetilde{x} - x_i$ is different from 0; there is no disutility if we use the actual value, so $d = d(\widetilde{x} - x_i)$ for an appropriate function $d(y)$, where $d(0) = 0$ and $d(y) > 0$ for $y \neq 0$.

The estimates are usually reasonably accurate, so the difference $x_i - \widetilde{x}$ is small, and we can expand the function $d(y)$ in Taylor series and keep only the first few terms in this expansion:

$$d(y) = d_0 + d_1 \cdot y + d_2 \cdot y^2 + \ldots$$

From $d(0) = 0$ we conclude that $d_0 = 0$. From $d(y) > 0$ for $y \neq 0$ we conclude that $d_1 = 0$ (else we would have $d(y) < 0$ for some small y) and $d_2 > 0$, so $d(y) = d_2 \cdot y^2 = d_2 \cdot (\widetilde{x} - x_i)^2$.

The overall disutility $d(\widetilde{x})$ of using $\widetilde{x}$ instead of each of the values $x_1, \ldots, x_n$ can be computed as the sum of the corresponding disutilities

$$d(\widetilde{x}) = \sum_{i=1}^{n} d(\widetilde{x} - x_i)^2 = d_2 \cdot \sum_{i=1}^{n} (\widetilde{x} - x_i)^2.$$

Maximizing utility $u(\widetilde{x}) \stackrel{\text{def}}{=} -d(\widetilde{x})$ is equivalent to minimizing disutility.

The resulting combined value. Since $d_2 > 0$, minimizing the disutility function is equivalent to minimizing the re-scaled disutility function

$$D(\widetilde{x}) \stackrel{\text{def}}{=} \frac{d(\widetilde{x})}{d_2} = \sum_{i=1}^{n} (\widetilde{x} - x_i)^2.$$

Differentiating this expression with respect to $\widetilde{x}$ and equating the derivative to 0, we get

$$\widetilde{x} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i.$$

This is the well-known sample mean.

2 Case of Interval Uncertainty: Formulation of the Problem

Formulation of the practical problem. In many practical situations, instead of the exact estimates x_i, we only know the intervals $[\underline{x}_i, \overline{x}_i]$ that contain the unknown values x_i. How do we select the value x in this case?

Towards precise formulation of the problem. For different values x_i from the corresponding intervals $[\underline{x}_i, \overline{x}_i]$, we get, in general, different values of utility

$$U(\widetilde{x}, x_1, \ldots, x_n) = -D(\widetilde{x}, x_1, \ldots, x_n),$$

where $D(\widetilde{x}, x_1, \ldots, x_n) = \sum_{i=1}^{n} (\widetilde{x} - x_i)^2$. Thus, all we know is that the actual (unknown) value of the utility belongs to the interval $[\underline{U}(\widetilde{x}), \overline{U}(\widetilde{x})] = [-\overline{D}(\widetilde{x}), -\underline{D}(\widetilde{x})]$, where

$$\underline{D}(\widetilde{x}) = \min D(\widetilde{x}, x_1, \ldots, x_n),$$

$$\overline{D}(\widetilde{x}) = \max D(\widetilde{x}, x_1, \ldots, x_n),$$

and min and max are taken over all possible combinations of values $x_i \in [\underline{x}_i, \overline{x}_i]$.

In such situations of interval uncertainty, decision making theory recommends using Hurwicz optimism-pessimism criterion [2–4], i.e., maximize the value

$$U(\widetilde{x}) \stackrel{\text{def}}{=} \alpha \cdot \overline{U}(\widetilde{x}) + (1 - \alpha) \cdot \underline{U}(\widetilde{x}),$$

where the parameter $\alpha \in [0, 1]$ describes the decision maker's degree of optimism. For $U = -D$, this is equivalent to minimizing the expression

$$D(\widetilde{x}) = -U(\widetilde{x}) = \alpha \cdot \underline{D}(\widetilde{x}) + (1 - \alpha) \cdot \overline{D}(\widetilde{x}).$$

What we do in this paper. In this paper, we describe an efficient algorithm for computing the value $\widetilde{x}$ that minimizes the resulting objective function $D(\widetilde{x})$.

3 Analysis of the Problem

Let us simplify the expressions for $\underline{D}(\widetilde{x})$, $\overline{D}(\widetilde{x})$, and $D(\widetilde{x})$. Each term $(\widetilde{x} - x_i)^2$ in the sum $D(\widetilde{x}, x_1, \ldots, x_n)$ depends only on its own variable x_i. Thus, with respect to x_i:

- the sum is the smallest when each of these terms is the smallest, and
- the sum is the largest when each term is the largest.

One can easily see that when x_i is in the $[\underline{x}_i, \overline{x}_i]$, the maximum of a term $(\widetilde{x} - x_i)^2$ is always attained at one of the interval's endpoints:

- at $x_i = \underline{x}_i$ when $\widetilde{x} \geq \widetilde{x}_i \stackrel{\text{def}}{=} \dfrac{\underline{x}_i + \overline{x}_i}{2}$ and
- at $x_i = \overline{x}_i$ when $\widetilde{x} < \widetilde{x}_i$.

Thus,

$$\overline{D}(\widetilde{x}) = \sum_{i:\widetilde{x} < \widetilde{x}_i} (\widetilde{x} - \overline{x}_i)^2 + \sum_{i:\widetilde{x} \geq \widetilde{x}_i} (\widetilde{x} - \underline{x}_i)^2.$$

Similarly, the minimum of the term $(\widetilde{x} - x_i)^2$ is attained:

- for $x_i = \widetilde{x}$ when $\widetilde{x} \in [\underline{x}_i, \overline{x}_i]$ (in this case, the minimum is 0);
- for $x_i = \underline{x}_i$ when $\widetilde{x} < \underline{x}_i$; and
- for $x_i = \overline{x}_i$ when $\widetilde{x} > \overline{x}_i$.

Thus,

$$\underline{D}(\widetilde{x}) = \sum_{i:\widetilde{x}>\overline{x}_i} (\widetilde{x} - \overline{x}_i)^2 + \sum_{i:\widetilde{x}<\underline{x}_i} (\widetilde{x} - \underline{x}_i)^2.$$

So, for $D(\widetilde{x}) = \alpha \cdot \underline{D}(\widetilde{x}) + (1-\alpha) \cdot \overline{D}(\widetilde{x})$, we get

$$D(\widetilde{x}) = \alpha \cdot \sum_{i:\widetilde{x}>\overline{x}_i} (\widetilde{x} - \overline{x}_i)^2 + \alpha \cdot \sum_{i:\widetilde{x}<\underline{x}_i} (\widetilde{x} - \underline{x}_i)^2 +$$

$$(1-\alpha) \cdot \sum_{i:\widetilde{x}<\widetilde{x}_i} (\widetilde{x} - \overline{x}_i)^2 + (1-\alpha) \cdot \sum_{i:\widetilde{x}\ge\widetilde{x}_i} (\widetilde{x} - \underline{x}_i)^2. \qquad (1)$$

Towards an algorithm. The presence or absence of different values in the above expression depends on the relation of $\widetilde{x}$ with respect to the values $\underline{x}_i$, $\overline{x}_i$, and $\widetilde{x}_i$. Thus, if we sort these $3n$ values into a sequence $s_1 \le s_2 \le \ldots \le s_{3n}$, then on each interval $[s_j, s_{j+1}]$, the function $D(\widetilde{x})$ is simply a quadratic function of $\widetilde{x}$.

A quadratic function attains its minimum on an interval either at one of its midpoints, or at a point when the derivative is equal to 0 (if this point is inside the given interval). Differentiating the above expression for $D(\widetilde{x})$, equating the derivative to 0, dividing both sides by 0, and moving terms proportional not containing $\widetilde{x}$ to the right-hand side, we conclude that

$$(\alpha \cdot \#\{i : \widetilde{x} < \underline{x}_i \text{ or } \widetilde{x} > \overline{x}_i\} + 1 - \alpha) \cdot \widetilde{x} =$$

$$\alpha \cdot \sum_{i:\widetilde{x}>\overline{x}_i} \overline{x}_i + \alpha \cdot \sum_{i:\widetilde{x}<\underline{x}_i} \underline{x}_i + (1-\alpha) \cdot \sum_{i:\widetilde{x}<\widetilde{x}_i} \overline{x}_i + (1-\alpha) \cdot \sum_{i:\widetilde{x}\ge\widetilde{x}_i} \underline{x}_i.$$

Since s_j is a listing of all thresholds values $\underline{x}_i$, $\overline{x}_i$, and $\widetilde{x}_i$, then for $\widetilde{x} \in (s_j, s_{j+1})$, the inequality $\widetilde{x} < \underline{x}_i$ is equivalent to $s_{j+1} \le \underline{x}_i$. Similarly, the inequality $\widetilde{x} > \underline{x}_i$ is equivalent to $s_j \ge \overline{x}_i$. In general, for values $\widetilde{x} \in (s_j, s_{j+1})$, the above equation gets the form

$$(\alpha \cdot \#\{i : \widetilde{x} < \underline{x}_i \text{ or } \widetilde{x} > \overline{x}_i\} + 1 - \alpha) \cdot \widetilde{x} =$$

$$\alpha \cdot \sum_{i:s_j\ge\overline{x}_i} \overline{x}_i + \alpha \cdot \sum_{i:s_{j+1}\le\underline{x}_i} \underline{x}_i + (1-\alpha) \cdot \sum_{i:s_{j+1}\le\widetilde{x}_i} \overline{x}_i + (1-\alpha) \cdot \sum_{i:s_j\ge\widetilde{x}_i} \underline{x}_i.$$

From this equation, we can easily find the desired expression for the value $\widetilde{x}$ at which the derivative is 0.

Thus, we arrive at the following algorithm.

4 Resulting Algorithm

First, for each interval $[\underline{x}_i, \overline{x}_i]$, we compute its midpoint $\widetilde{x}_i = \dfrac{\underline{x}_i + \overline{x}_i}{2}$. Then, we sort the $3n$ values $\underline{x}_i$, $\overline{x}_i$, and $\widetilde{x}_i$ into an increasing sequence $s_1 \le s_2 \le \ldots \le s_{3n}$. To cover the whole real line, to these values, we add $s_0 = -\infty$ and $s_{3n+1} = +\infty$.

We compute the value of the objective function (1) on each of the endpoints $s_1, \ldots, s_{3n}$. Then, for each interval (s_i, s_{j+1}), we compute the value

$$\widetilde{x} = \frac{\alpha \cdot \sum\limits_{i: s_j \geq \overline{x}_i} \overline{x}_i + \alpha \cdot \sum\limits_{i: s_{j+1} \leq \underline{x}_i} \underline{x}_i + (1-\alpha) \cdot \sum\limits_{i: s_{j+1} \leq \widetilde{x}_i} \overline{x}_i + (1-\alpha) \cdot \sum\limits_{i: s_j \geq \widetilde{x}_i} \underline{x}_i}{\alpha \cdot \#\{i : \widetilde{x} < \underline{x}_i \text{ or } \widetilde{x} > \overline{x}_i\} + 1 - \alpha}.$$

If the resulting value $\widetilde{x}$ is within the interval (s_i, s_{j+1}), we compute the value of the objective function (1) corresponding to this $\widetilde{x}$.

After that, out of all the values $\widetilde{x}$ for which we have computed the value of the objective function (1), we return the value $\widetilde{x}$ for which objective function $D(\widetilde{x})$ was the smallest.

What is the computational complexity of this algorithm. Sorting $3n = O(n)$ values $\underline{x}_i$, $\overline{x}_i$, and $\widetilde{x}_i$ takes time $O(n \cdot \ln(n))$.

Computing each value $D(\widetilde{x})$ of the objective function requires $O(n)$ computational steps. We compute $D(\widetilde{x})$ for $3n$ endpoints and for $\leq 3n + 1$ values at which the derivative is 0 at each of the intervals (s_j, s_{j+1}) – for the total of $O(n)$ values.

Thus, overall, we need $O(n \cdot \ln(n)) + O(n) \cdot O(n) = O(n^2)$ computation steps. Hence, our algorithm runs in quadratic time.

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Plans Are Worthless but Planning Is Everything: A Theoretical Explanation of Eisenhower's Observation

Angel F. Garcia Contreras, Martine Ceberio, and Vladik Kreinovich

Department of Computer Science, University of Texas at El Paso
El Paso, TX 79968, USA
afgarciacontreras@miners.utep.edu,
mceberio@utep.edu, vladik@utep.edu

Abstract. The 1953-1961 US President Dwight D. Eisenhower emphasized that his experience as the Supreme Commander of the Allied Expeditionary Forces in Europe during the Second World War taught him that "plans are worthless, but planning is everything". This sound contradictory: if plans are worthless, why bother with planning at all? In this paper, we show that Eisenhower's observation has a meaning: while directly following the original plan in constantly changing circumstances is often not a good idea, the existence of a pre-computed original plan enables us to produce an almost-optimal strategy – a strategy that would have been computationally difficult to produce on a short notice without the pre-existing plan.

1 Introduction: Eisenhower's Seemingly Paradoxical Observation

Eisenhower's observation. Dwight D. Eisenhower, the Supreme Commander of the Allied Expeditionary Forces in Europe during the Second World War and later the US President, emphasized that his war experience taught him that "plans are worthless, but planning is everything"; see, e.g., [1].

At first glance, this observation seems paradoxical. At first glance, the Eisenhower's observation sounds paradoxical: if plans are worthless, why bother with planning at all?

What we do in this paper. In this paper, we show that this Eisenhower's observation has a meaning. Namely, it means that:

- while following the original plan in constantly changing circumstances is often not a good idea,
- the existence of a pre-computed original plan enables us to produce an almost-optimal strategy (a strategy that would have been computationally difficult to produce on a short notice without the pre-existing plan).

2 Analysis of the Problem

Rational decision making: a brief reminder. According to decision making theory, decisions by a rational decision maker can be described as maximize the value a certain function known as utility; see, e.g., [3, 4]. In financial situations, when a company needs to make a decision, the overall profit can be used as the utility value; in more complex situations, the utility function combines different aspects of gain and loss related to different decisions.

Let us describe this in precise terms. Let x denote a possible action, a describes the situation, and let $u(x, a)$ denote the utility that results from performing action x in situation a.

To describe a possible action, we usually need to describe the values of several different quantities. For example, a decision about a plant involves selecting amount of gadgets of different type manufactured at this plant – and maybe also the parameters characterizing these gadgets. Let us denote the parameters describing an action by $x_1, \ldots, x_n$. In these terms, an action can be characterized by the tuple $x = (x_1, \ldots, x_n)$.

Similarly, in general, we need several different quantities to describe a situation, so we will describe a situation by a tuple $a = (a_1, \ldots, a_m)$.

In these terms, what is planning. Let $\widetilde{a}$ describe the original situation. Based on this situation, we come up with an action $\widetilde{x}$ that maximizes the corresponding utility: $u(\widetilde{x}, \widetilde{a}) = \max\limits_{x} u(x, \widetilde{a})$. Computing this optimal action $\widetilde{x}$ is what we usually call *planning*.

Situations change. At the moment when we need to start acting, the situation may have changed in comparison with the original situation $\widetilde{a}$, to a somewhat different situation a. Let us denote the corresponding change by $\Delta a \stackrel{\text{def}}{=} a - \widetilde{a}$. In terms of this difference, the new situation takes the form $a = \widetilde{a} + \Delta$.

A not-always-very-good option: applying the original plan to the new situation. One possibility is to simply ignore the change, and apply the original plan $\widetilde{x}$ – which was optimal for the original situation $\widetilde{a}$ – to the new situation $a = \widetilde{a} + \Delta a$.

This plan is, in general, not optimal for the new situation. Thus, in comparison to the actually optimal plan x^{opt} for which

$$u(x^{\text{opt}}, \widetilde{a} + \Delta a) = \max_{x} u(x, \widetilde{a} + \Delta a),$$

we lose the amount $L_0 \stackrel{\text{def}}{=} u(x^{\text{opt}}, \widetilde{a} + \Delta a) - u(\widetilde{x}, \widetilde{a} + \Delta a)$.

A better option: trying to modify the original plan. Why cannot we just find the optimal solution for the new situation? Because optimization is, in general, an NP-hard problem (see, e.g., [2, 5]), meaning that it is not possible to find the exact optimum in reasonable time.

What we can do is try to use some feasible algorithm – e.g., solving a system of linear equations – to replace the original plan $\widetilde{x}$ with a modified plan $\widetilde{x} + \Delta x$.

Due to NP-hardness, this feasibly modified plan is, in general, not optimal, but we hope that the resulting loss $L_1 \stackrel{\text{def}}{=} u(x^{\text{opt}}, \widetilde{a} + \Delta a) - u(\widetilde{x} + \Delta x, \widetilde{a} + \Delta a)$ is much smaller than the loss L_0 corresponding to the use of the original plan $\widetilde{x}$.

What we do in this paper. In this paper, we analyze the values of both losses and we show that indeed, L_1 is much smaller than L_0. So, in many situations, even if the loss L_0 is so large that the corresponding strategy (of directly using the original plan) is worthless, the modified plan may leads to a reasonably small loss $L_1 \ll L_0$ – thus explaining Eisenhower's observation.

Estimating L_0. We assume that the difference Δa is reasonably small, so the corresponding difference in action $\Delta x^{\text{opt}} \stackrel{\text{def}}{=} x^{\text{opt}} - \widetilde{x}$ is also small. We can therefore expand the expression for the loss L_0 in Taylor series and keep only terms which are linear and quadratic with respect to Δx. Thus, we get

$$L_0 = u(x^{\text{opt}}, \widetilde{a} + \Delta a) - u(x^{\text{opt}} - \Delta x^{\text{opt}}, \widetilde{a} + \Delta a) =$$

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x^{\text{opt}}, \widetilde{a} + \Delta a) \cdot \Delta x_i^{\text{opt}} +$$

$$\frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{i'=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_{i'}}(x^{\text{opt}}, \widetilde{a} + \Delta a) \cdot \Delta x_i^{\text{opt}} \cdot \Delta x_{i'}^{\text{opt}} + o((\Delta a)^2).$$

By definition, the action x^{opt} maximizes the utility $u(x, \widetilde{a} + \Delta a)$. Thus, we have $\dfrac{\partial u}{\partial x_i}(x^{\text{opt}}, \widetilde{a} + \Delta a) = 0$, and the above expression for the loss L_0 takes the simplified form

$$L_0 = \frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{i'=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_{i'}}(x^{\text{opt}}, \widetilde{a} + \Delta a) \cdot \Delta x_i^{\text{opt}} \cdot \Delta x_{i'}^{\text{opt}} + o((\Delta a)^2). \qquad (1)$$

The values Δx_i^{opt} can be estimated from the above condition

$$\frac{\partial u}{\partial x_i}(x^{\text{opt}}, \widetilde{a} + \Delta a) = \frac{\partial u}{\partial x_i}(\widetilde{x} + \Delta x^{\text{opt}}, \widetilde{a} + \Delta) = 0.$$

Expanding this expression in Taylor series in terms of Δx_i and Δa_j and taking into account that $\dfrac{\partial u}{\partial x_i}(\widetilde{x}, \widetilde{a}) = 0$ (since for $a = \widetilde{a}$, the utility is maximized by the action $x = \widetilde{x}$), we conclude that for every i, we have

$$\sum_{i'=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_{i'}}(\widetilde{x}, \widetilde{a}) \cdot \Delta x_{i'}^{\text{opt}} + \sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_i \partial a_j}(\widetilde{x}, \widetilde{a}) \cdot \Delta a_j + o(\Delta x, \Delta a) = 0.$$

Thus, the first approximation Δx_i to the values Δx_i^{opt} can be determined as a solution to a system of linear equations:

$$\sum_{i'=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_{i'}}(\widetilde{x}, \widetilde{a}) \cdot \Delta x_j = -\sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_i \partial a_j}(\widetilde{x}, \widetilde{a}) \cdot \Delta a_j. \qquad (2)$$

A solution to a system of linear equations is a linear combination of the right-hand sides. Thus, the values Δx_i are a linear function of Δa_j. Substituting these linear expressions into the formula (1), we conclude that *the loss L_0 is a quadratic function of Δa_j*, i.e., that $L_0 = \sum\limits_{j=1}^{m} \sum\limits_{j'=1}^{m} k_{jj'} \cdot \Delta a_j \cdot \Delta a_{j'} + o((\Delta a)^2)$ for some coefficients $k_{jj'}$.

Estimating L_1. In the previous section, we considered what happens if we use the original plan $\widetilde{x}$ – which was optimal in the original situation $\widetilde{a}$ – in the changed situation $a = \widetilde{a} + \Delta a$. Since the original plan is optimal only for the original situation, but not for the new one, using this not-optimal plan leads to the loss L_0, a loss which we estimated as being quadratic in terms of Δa.

To decrease this loss, we need to update the action x. As we have already mentioned, exactly computing the optimal action $x^{\rm opt}$ is, in general, an NP-hard – i.e., computationally intractable – problem. However, as we have also mentioned, the first approximation Δx_i to the desired difference $\Delta x^{\rm opt}$ – and thus, the first approximation to the newly optimal solution $x^{\rm opt}$ – can be obtained by solving a system of linear equations (2).

The system (2) of linear equations is feasible to solve. Thus, it is reasonable to consider using the action $x^{\rm lin} = \widetilde{x} + \Delta x$ instead of the original action $\widetilde{x}$. Let us estimate how much we lose if we use this new action $x^{\rm lin}$ instead of the optimal action $x_i^{\rm opt}$.

The fact that the difference Δx is the first approximation to the optimal difference $\Delta x^{\rm opt}$ means that we can write $\Delta x^{\rm opt} = \Delta x + \delta x$, where the remaining term $\delta x \stackrel{\rm def}{=} \Delta x^{\rm opt} - \Delta x = x^{\rm opt} - x^{\rm lin}$ is of second order in terms of Δx and Δa: $\delta x = O((\Delta x)^2, (\Delta a)^2)$. Since in the first approximation, Δx has the same order as Δa, we thus get $\delta x = O((\Delta a)^2)$.

The loss L_1 of using $x^{\rm lin} = x^{\rm opt} - \delta x$ instead of $x^{\rm opt}$ is equal to the difference

$$L_1 = u(x^{\rm opt}, \widetilde{a} + \Delta a) - u(x^{\rm lin}, \widetilde{a} + \Delta a) = u(x^{\rm opt}, \widetilde{a} + \Delta a) - u(x^{\rm opt} - \delta x, \widetilde{a} + \Delta a).$$

If we expand this expression in δx and keep only linear and quadratic terms, we conclude that

$$L_1 = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x^{\rm opt}, \widetilde{a} + \Delta a) \cdot \delta x_i +$$

$$\frac{1}{2} \cdot \sum_{i=1}^{n} \sum_{i'=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_{i'}}(x^{\rm opt}, \widetilde{a} + \Delta a) \cdot \delta x_i \cdot \delta x_{i'} + o((\delta x)^2).$$

Since $x^{\rm opt}$ is the action that, for $a = \widetilde{a} + \Delta a$, maximizes utility, we get

$$\frac{\partial u}{\partial x_i}(x^{\rm opt}, \widetilde{a} + \Delta a) = 0.$$

Thus, the expression for L_1 gets a simplified form

$$L_1 = \frac{1}{2} \cdot \sum_{i-1}^{n} \sum_{i'=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_{i'}}(x^{\rm opt}, \widetilde{a} + \Delta a) \cdot \delta x_i \cdot \delta x_{i'} + o((\delta x)^2).$$

We know that the values δx_i are quadratic in Δa; thus, we conclude that for the modified action, *the loss L_1 is a 4-th order function of Δa_j*, i.e., that

$$L_1 = \sum_{j=1}^{m} \sum_{j'=1}^{m} \sum_{j''=1}^{m} \sum_{j'''=1}^{m} k_{jj'j''j'''} \cdot \Delta a_j \cdot \Delta a_{j'} \cdot \Delta a_{j''} \cdot \Delta a_{j'''} + o((\Delta a)^5)$$

for some coefficients $k_{jj'j''j'''}$.

3 Conclusions

We conclude that:

- the loss L_0 related to using the original plan is quadratic in Δa, while
- the loss L_1 related to using a feasibly modified plan is of 4th order in terms of Δa.

For reasonably small Δa, we have $L_1 \sim (\Delta a)^4 \ll L_0 \sim (\Delta a)^2$.

Let $\varepsilon > 0$ be the maximum loss that we tolerate. Since $L_1 \ll L_0$, we have three possible cases: (1) $\varepsilon < L_1$, (2) $L_1 \le \varepsilon \le L_0$, and (3) $L_0 < \varepsilon$. In the first case, even using the modified action does not help. In the third case, the change in the situation is so small that it is Ok to use the original plan $\widetilde{x}$.

In the second case, we have exactly the Eisenhower situation:

- if we use the original plan $\widetilde{x}$, the resulting loss L_0 much larger than we can tolerate; in this sense, the original plan is worthless;
- on the other hand, if we feasible modify the original plan into x^{lin}, then we get an acceptable action.

So, we indeed get a theoretical justification of Eisenhower's observation.

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Short Presentations

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Reliable Evaluation of the L_2-Norm of a Stable Linear Filter Using Interval Constraints Solving Techniques

Leobardo Valera and Martine Ceberio

Department of Computer Science, University of Texas at El Paso
El Paso, TX 79968, USA
{lvalera, mceberio}@utep.edu

Abstract. Nowadays, digital signals (DS) are everywhere, from the music we listen to on our portable devices, to communication between security agencies. All of these are made possible thanks to Digital Signals [1]. Occasionally, DS are not clear enough and some information might be lost. We all have experienced this phenomenon while on a phone call for instance, when we cannot hear our interlocutor anymore.

Noise in Digital Signal can be removed or at least minimized using digital filters (W). Such filters satisfy the Lyapunov equation and their L_2-norm is used to estimate how changes in the precision of the input signal can affect the output.

In this talk, we will show how we used Interval Constraint Solving Techniques (ICST) [4–6] to implement a reliable version of efficient Hammarling's algorithm [2, 3] to solve the Lyapunov equation, obtain the coefficients of the filter, and compute its L_2 norm.

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Parameter Estimation of a Dynamic System Using VSPODE

Angel Garcia, Leobardo Valera, and Martine Ceberio

Department of Computer Science, University of Texas at El Paso
El Paso, TX 79968, USA
afgarciacontreras@miners.utep.edu, {lvalera, mceberio}@utep.edu

Abstract. In initial value problems for ODE's with uncertainty on some parameters and/or initial conditions, much work has been done to find an enclosure of all possible solutions to the ODE system [2]. However, less work has been done in the inverse problem, which consists in estimating the initial conditions of a phenomenon whose observations are available up to some time t (diagnosis), to then be able to reconstruct its behavior, beyond t (prognosis) [3].
In this talk, we use VSPODE (Validation Solver for Parametric ODE's) [1], which uses automatic differentiation and interval Taylor expansions to handle uncertainty in the form of intervals [4] and then estimate the initial conditions of a dynamic system to prevent a future undesirable outcome.

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Reliable-Eig: An Algorithm to Compute the Spectrum of a Non-Defective Matrix Using Interval Constraints Solving Techniques

Leobardo Valera, Jesus Padilla, and Martine Ceberio

Department of Computer Science, University of Texas at El Paso
El Paso, TX 79968, USA
jjpadillamendez@miners.utep.edu, {lvalera, mceberio}@utep.edu

Abstract. Knowing the spectrum of a matrix is essential in many areas of engineering and sciences. For example, 1/ the spectrum of a matrix is used to write the solution of a linear system of ordinary differential equations allowing us to know the behavior of a mass-spring system, or the intensity of an RCL circuit [4]. 2/ It is also used to compute the L_2-norm of a digital filter, which is important because help us to figure out how small changes in the input signal could affect the whole input-output system [1–3].
Computing the spectrum of a matrix in a reliable way has been a challenge since many years ago. This is because there is not a direct way to compute the eigenvalues of matrix. This fact is a consequence of Abel-Galois's theorem that was proved in 1824 that there is not an algebraic formula to obtain the roots of a polynomial of degree greater than four. This means that any algorithm to compute the spectrum of a matrix must be iterative.
Iterative algorithms are not reliable due to the finite representation of real numbers as floating points, and the errors caused by rounding are propagated with each iteration.
In this talk, we propose to use Interval Constraint Solving Techniques (ICST) [5–7] to compute in a reliable way the eigenvalues and eigenvectors of a non-defective matrix.

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Authors