A comparison of some T-norms and T-conorms over the steady state of a fuzzy Markov chain

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Abstract. This paper shows a comparison of several T–norms and T–conorms used to compute the steady state of a fuzzy Markov chain. The effect of every of the selected norms over the mean and variance of a fuzzy Markov chain is evaluated using some simulations. Some recommendations and concluding remarks are given.

Keywords: Fuzzy T-norms, Fuzzy T-conorms, Fuzzy Markov chains

1 Motivation

Fuzzy Markov chains is one of the most popular fuzzy stochastic process (see Sanchez [7,8], Avrachenkov & Sanchez [3,2], and Araiza, Xiang, Kosheleva & Skülj [1]) so there are different algorithms, fuzzy relations and compositions to compute the stable state possibilities of the fuzzy process.

Most of published works on fuzzy Markov chains apply the $\max - \min$ composition to obtain the steady state of the process (see Figueroa-García [4,5]), so there is the possibility of using other compositions to see the effect over its steady state.

It is well known that different T–norms and T–conorms lead to closer/wider results in fuzzy inference systems, so we want to evaluate the effect of several norms such as Gödel, Archimedean, Łukasiewicz, Hamacher, Sugeno–Weber, and Yager T–norms (see Klement et al. [6]) over the mean/variance of a fuzzy Markov chain (a.k.a weak inference of a stochastic process).

2 Fuzzy Markov chains

Let $\mathcal{P}(X)$ be the class of all crisp sets, $\mathcal{F}(X)$ is the class of all fuzzy sets, $\mathcal{F}_1(X)$ is the class of all convex fuzzy sets, and $I = [0, 1]$ be the set of values in the unit interval. A fuzzy set namely A is characterized by a membership function $\mu_A : X \to I$ defined over a universe of discourse $x \in X$. Thus, a fuzzy set A is the set of ordered pairs $x \in X$ and its membership degree, $\mu_A(x)$, i.e.,

$$A = \{(x, \mu_A(x)) \,|\, x \in X\}. \quad (1)$$

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Let us denote $\mathcal{F}_1(\mathbb{R})$ as the class of all fuzzy numbers. Avrachenkov & Sanchez (see [3]) defined a fuzzy Markov chains as follows.

Definition 1 *Let $S = \{1, 2, \cdots, n\}$, then a finite fuzzy set or a fuzzy distribution on S is defined by a map $x : S \to [0, 1]$ represented by a vector $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ where $0 \leqslant x_i \leqslant 1$, $i \in S$. The set of all fuzzy sets is denoted by $\mathcal{F}(S)$.*

A fuzzy relation matrix P on the cartesian product $S \times S$ where P is defined by a matrix $\{p_{ij}\}_{i,j=1}^{n}$, with $0 \leqslant p_{ij} \leqslant 1$, $i, j \in S$. This fuzzy relationship matrix P allows us to define the transition matrix of the m states of the Markov chain between each time instant t in the following form:

Definition 2 *At each instant t, $t = 1, 2, \cdots, n$, the state of the stochastic process is described by the fuzzy set $x^{(t)} \in \mathcal{F}(S)$. The transition law of the Markov chain is given by the fuzzy relation P as follows, at the instant t, $T = 1, 2, \cdots, n$.*

$$\mathbf{x}^{(t+1)} = \bigvee_{i \in S} \{\mathbf{x}^{(t)} \wedge p_{ij}\}, \ j \in S. \tag{2}$$

were i and j are the initial and final state of the transition $i, j = 1, 2, \cdots, m$ and $\mathbf{x}^{(0)}$ is the initial fuzzy set or the initial distribution of the process.

Like in probabilistic Markov chains, the convergence of the chain in a fuzzy environment can be found using a time limiting criterion, so the powers of a fuzzy transition matrix P are:

$$p_{ij}^{t} \triangleq \bigvee_{k \in S} \{p_{ik} \wedge p_{kj}^{t-1}\} \tag{3}$$

or in a matrix form:

$$P^{t} \triangleq P \circ P^{t-1} \tag{4}$$

where $\circ$ denotes a fuzzy composition, and the limiting distribution of P is:

Definition 3 *Let the powers of the fuzzy transition matrix converge in τ steps to a non periodic solution, then the associated fuzzy Markov chain is called non periodic (or aperiodic) and $P^{*} = P^{\tau}$ is called a limiting fuzzy transition matrix.*

Then the main idea is to evaluate the effect of several T–norms and T–conorms over the expected value and variance of P^{τ}, as defined as follows:

$$E(P) = \frac{\sum_i i \cdot \check{x}_i}{\sum_i \check{x}_i}, \tag{5}$$

$$V(P) = \frac{\sum_i (i - E(P))^2 \cdot \check{x}_i}{n - 1}, \tag{6}$$

where i is the initial distribution of P and $\check{x}_i$ is the i_{th} element of the limiting distribution of P^{τ}.

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