P-completeness of testing solutions of parametric interval linear systems

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Abstract. We deal with parametric interval linear equations and its particular sub-classes defined by symmetry of the constraint matrix. We show that the problem of checking whether a given vector is a solution is a P-complete problem, meaning that there unlikely exists a polynomial closed form arithmetic formula of Oettli–Prager-type describing the solution set. We leave as an open problem whether P-completeness concerns also the simplest version of the symmetric solution set.

Keywords: Interval computation, parametric system, linear system, P-complete problem.

1 Introduction

Let us introduce some notation first. An interval matrix is defined as

$$\boldsymbol{A} := \{A \in \mathbb{R}^{m\times n};\ \underline{A} \le A \le \overline{A}\},$$

where $\underline{A}$ and $\overline{A}$, $\underline{A} \le \overline{A}$, are given matrices and the inequality is understood entrywise. The midpoint and radius matrices are defined as

$$A^c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A^\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all interval $m \times n$ matrices is denoted by $\mathbb{IR}^{m\times n}$. Interval vectors are defined analogously.

In this paper, we deal with solutions of interval linear systems. Consider a parametric interval linear system

$$A(p)x = b(p),$$

in which parameters have a linear structure

$$A(p) = \sum_{k=1}^{K} A^{(k)} p_k, \quad b(p) = \sum_{k=1}^{K} b^{(k)} p_k.$$

Herein, $A^{(1)}, \dots, A^{(K)} \in \mathbb{R}^{n\times n}$ and $b^{(1)}, \dots, b^{(K)} \in \mathbb{R}^n$ are fixed, and the parameters $p_1, \dots, p_K$ come from their respective interval domains $\boldsymbol{p}_1, \dots, \boldsymbol{p}_K \in \mathbb{IR}$.

The solution set of this parametric system is denoted by Σ and consists of all the solutions of all the linear systems corresponding to all possible combinations of parameters $p \in \boldsymbol{p}$, that is,

$$\Sigma = \{x \in \mathbb{R}^n;\ \exists p \in \boldsymbol{p} : A(p)x = b(p)\}.$$

As an important subclass, we will also consider a symmetric interval system, where the dependencies determine the constraint matrix to be symmetric. Given $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{p} \in \mathbb{IR}^K$, the symmetric solution set reads

$$\Sigma_{sym} = \{x \in \mathbb{R}^n;\ \exists p \in \boldsymbol{p}\, \exists A \in \boldsymbol{A} : Ax = b(p),\ A = A^T\}. \tag{1}$$

If there are no dependencies in the right-hand side, corresponding symmetric solution set is

$$\Sigma^*_{sym} = \{x \in \mathbb{R}^n;\ \exists A \in \boldsymbol{A}\, \exists b \in \boldsymbol{b} : Ax = b,\ A = A^T\}, \tag{2}$$

where $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{IR}^n$.

Problem formulation. Given $x^* \in \mathbb{R}^n$, decide whether $x^* \in \Sigma$, or $x^* \in \Sigma_{sym}$, or $x^* \in \Sigma^*_{sym}$.

This problem can be solved by means of linear programming simply considering interval parameters as variables. The decision problems then reduce to checking feasibility of the linear systems

$$A(p)x^* = b(p),\ p \in \boldsymbol{p}$$

with respect to variables p, and

$$Ax^* = b(p),\ A = A^T,\ A \in \boldsymbol{A},\ p \in \boldsymbol{p}$$

with respect to variables A and p.

The solution set of the standard interval linear system $Ax = b$, $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ is defined as

$$\{x \in \mathbb{R}^n;\ \exists A \in \boldsymbol{A}\, \exists b \in \boldsymbol{b} : Ax = b, \},$$

and characterized by the Oettli–Prager inequalities [9]

$$|A^c x - b^c| \le A^\Delta |x| + b^\Delta.$$

Obviously, checking whether a given point $x^* \in \mathbb{R}^n$ satisfies this system is in NC, which is the class of decision problems decidable in polylogarithmic time on a parallel computer with a polynomial number of processors. Thus, unless P=NC, it is not a P-hard problem.

Since 1990s, there were various attempts to find also a closed form arithmetic description of Σ^*_{sym} and the related solution sets [1–3]. The first approaches utilized the Fourier–Motzkin elimination (yielding possibly double exponential number of constraints). In [5], there was presented an explicit description consisting of an exponential number of constraints, and later improvements include

[6–8]. For a general linear parametric solution set Σ, an explicit description was presented in [10].

So far, no explicit polynomial Oettli–Prager-type characterization of the solutions of the parametric systems (or some particular subclasses such as the symmetric case) was found. The main message of this paper is to show that such a characterization unlikely exists because the problem is P-complete.

2 Results

Now, we state and prove our main results on P-completeness of testing solutions of parametric systems. We will utilize the fact that checking solvability of a linear system $Ax = b$, $x \geq 0$ is P-complete under NC-reduction (i.e., in polylogarithmic time on a parallel computer with a polynomial number of processors); see, e.g., Greenlaw et al. [4].

First, we state the result for a general parametric system, and the we strengthen it to the symmetric one.

Theorem 1. *Checking $x^* \in \Sigma$ is a P-complete problem.*

Theorem 2. *Checking $x^* \in \Sigma_{sym}$ is a P-complete problem.*

We did not succeeded in our attempts to prove P-completeness of testing $x^* \in \Sigma^*_{sym}$, which remains as an open problem:

Open problem: Is the problem of checking $x^* \in \Sigma^*_{sym}$ P-complete?

Even though complexity of checking $x^* \in \Sigma^*_{sym}$ is unknown, the problem of finding the best "certificate" is P-complete. Herein, "the best" means that it maximizes the given linear function $\mathrm{tr}(AF)$, where $\mathrm{tr}(\cdot)$ stands for the matrix trace.

Theorem 3. *The following problem is P-complete: Given $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{b} \in \mathbb{IR}^n$, symmetric $F \in \mathbb{R}^{n\times n}$, and $x^* \in \mathbb{R}^n$, among symmetric matrices $A \in \boldsymbol{A}$ for which $Ax^* \in \boldsymbol{b}$, find the matrix for which $\mathrm{tr}(AF)$ is the largest possible.*

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