Why Burgers Equation: Symmetry-Based Approach

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Abstract. In many application areas ranging from shock waves to acoustics, we encounter the same partial differential equation known as the Burgers' equation. The fact that the same equation appears in different application domains, with different physics, makes us conjecture that it can be derived from the fundamental principles. Indeed, in this paper, we show that this equation can be uniquely determined by the corresponding symmetries.

1 Formulation of the Problem

Burgers' equation is ubiquitous. In many application areas ranging from fluid dynamics to nonlinear acoustics, gas dynamics, and dynamics of traffic flows, we encounter the Burgers' equation; see, e.g., [4, 5]:

$$\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = d \cdot \frac{\partial^2 u}{\partial x^2}. \qquad (1)$$

In particular, our interest in this equation comes from the use of these equations for describing shock waves; see, e.g., [2, 3].

Is there a common explanation for this empirical ubiquity? The fact that the Burgers' equation naturally appears in many different areas seems to indicate that this equation reflects some fundamental ideas, and not just ideas related to liquid or gas dynamics.

What we do in this paper. In this paper, we show that indeed, the Burgers' equation can be determined from fundamental principles.

2 Let Us Use Symmetries

Why symmetries. How do we make predictions in general? We observe that, in several situations, a body left in the air fell down. We thus conclude that in similar situations, a body will also fall down. Behind this conclusion is the fact that there is some similarity between the new and the old situations. In other words, there are transformations that transform the old situation into a new one – under which the physics will be mostly preserved, i.e., which form what physicists call *symmetries*.

In the falling down example, we can move to a new location, we can rotate around – the falling process will remain. Thus, shifts and rotations are symmetries of the falling-down phenomena.

In more complex situations, the behavior of a system may change with shift or with rotation, but the equations describing such behavior remain the same.

So, let us use symmetries. Symmetries are the fundamental reason why we are capable of predictions. Not surprisingly, symmetries have become one of the main tools of modern physics; see, e.g., [1]. Let us therefore use symmetries to explain the ubiquity of the Burgers' equation.

Which symmetries should we use. Numerical values of physical quantities depend on the measuring unit. For example, when we measure distance x first in meters and then in centimeters, the quantity remains the same, but it numerical values change: instead of the original value x, we get $x' = \lambda \cdot x$ for $\lambda = 100$.

In many cases, there is no physically selected unit of length. In such cases, it is reasonable to require that the corresponding physical equations be invariant with respect to such change of measuring unit, i.e., with respect to the transformation $x \to x' = \lambda \cdot x$.

Of course, once we change the unit for measuring x, we may need to change related units. For example, if we change a unit of current I in Ohm's formula $V = I \cdot R$, for the equation to remain valid we need to also appropriately change, e.g., the unit in which we measure voltage V.

In our case, there seems to be no preferred measuring unit, so it is reasonable to require that the corresponding equation be invariant under transformations $x \to \lambda \cdot x$ if we appropriately change measuring units for all other quantities.

3 What Are the Symmetries of the Burgers' Equation

We want to check if, for every λ, once we combine the re-scaling $x \to x' = \lambda \cdot x$ with the appropriate re-scalings $t \to t' = a(\lambda) \cdot t$ and $u \to u' = b(\lambda) \cdot u$, for some $a(\lambda)$ and $b(\lambda)$, the Burgers' equation (1) will preserve its form.

By keeping only the time derivative in the left-hand side of the equation, we get an equivalent form of the Burgers' equation in which this time derivative is described as a function on the current values of u:

$$\frac{\partial u}{\partial t} = -u \cdot \frac{\partial u}{\partial x} + d \cdot \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

After the transformation, e.g., the partial derivative $\dfrac{\partial u}{\partial t}$ is multiplied by $\dfrac{b(\lambda)}{a(\lambda)}$:

$$\frac{\partial u'}{\partial t'} = \frac{b(\lambda)}{a(\lambda)} \cdot \frac{\partial u}{\partial t},$$

and, more generally, the equation (2) gets transformed into the following form:

$$\frac{b(\lambda)}{a(\lambda)} \cdot \frac{\partial u}{\partial t} = -b(\lambda) \cdot \frac{b(\lambda)}{\lambda} \cdot u \cdot \frac{\partial u}{\partial x} + d \cdot \frac{b(\lambda)}{\lambda^2} \cdot \frac{\partial^2 u}{\partial x^2}. \tag{3}$$

Dividing both sides of this equation by the coefficient $\dfrac{b(\lambda)}{a(\lambda)}$ at the time derivative, we conclude that

$$\frac{\partial u}{\partial t} = -\frac{b(\lambda) \cdot a(\lambda)}{\lambda} \cdot u \cdot \frac{\partial u}{\partial x} + d \cdot \frac{a(\lambda)}{\lambda^2} \cdot \frac{\partial^2 u}{\partial x^2}. \tag{4}$$

By comparing the equations (2) and (4), we conclude that they are equivalent if the coefficients at the two terms in the right-hand side are the same, i.e., if $\dfrac{b(\lambda) \cdot a(\lambda)}{\lambda} = 1$ and $\dfrac{a(\lambda)}{\lambda^2} = 1$. The second equality implies that $a(\lambda) = \lambda^2$, and the first one, that $b(\lambda) = \dfrac{\lambda}{a(\lambda)} = \lambda^{-1}$.

Thus, the Burgers' equation is invariant under the transformation $x \to \lambda \cdot x$, $t \to \lambda^2 \cdot t$, and $u \to \lambda^{-1} \cdot u$.

4 Burgers' Equation Can Be Uniquely Determined by Its Symmetries

Formulation of the problem. Let us consider a general equation in which the time derivative of u depends on the current values of u:

$$\frac{\partial u}{\partial t} = f\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right). \tag{5}$$

Here, we assume that the function f is analytical, i.e., that it can be expanded into Taylor series

$$f\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) =$$

$$\sum_{i_0, i_1, \ldots, i_k} a_{i_1 \ldots i_k} \cdot u^{i_0} \cdot \left(\frac{\partial u}{\partial x}\right)^{i_1} \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)^{i_2} \cdot \ldots \cdot \left(\frac{\partial^k u}{\partial x^k}\right)^{i_k}, \tag{6}$$

where $i_0, i_1, \ldots, i_k$ are non-negative integers.

We are looking for all possible cases in which this equation is invariant under the transformation $x \to \lambda \cdot x$, $t \to \lambda^2 \cdot t$, and $u \to \lambda^{-1} \cdot u$.

Analysis of the problem. Under the above transformation, the left-hand side of the equation (5) is multiplied by $\dfrac{\lambda^{-1}}{\lambda^2} = \lambda^{-3}$. On the other hand, each term in the expansion (6) of the right-hand side of the formula (5) is multiplied by

$$\left(\lambda^{-1}\right)^{i_0} \cdot \left(\lambda^{-2}\right)^{i_1} \cdot \left(\lambda^{-3}\right)^{i_2} \cdot \ldots \cdot \left(\lambda^{-(k+1)}\right)^{i_k}, \tag{7}$$

i.e., by λ^{-D}, where we denoted

$$D = i_0 + 2 \cdot i_1 + 3 \cdot i_2 + \ldots + (k+1) \cdot i_k. \tag{8}$$

The equation is invariant if the left-hand side and right-hand side are multiplied by the same coefficient, i.e., if $D = 3$. Thus, in the invariant case, we can have only terms for which

$$i_0 + 2 \cdot i_1 + 3 \cdot i_2 + \ldots + (k+1) \cdot i_k = 3. \tag{9}$$

Here, the values $i_0, \ldots, i_k$ are non-negative integers. So, if we had $i_j > 0$ for some $j \geq 3$, i.e., $i_j \geq 1$, the left-hand side of the formula (9) would be greater than or equal to $j+1 \geq 4$, so it cannot be equal to 3. Thus, in the invariant case, we can only have values i_0, i_1, and i_2 possibly different from 0. In this case, the formula (9) takes a simplified form

$$i_0 + 2 \cdot i_1 + 3 \cdot i_2 = 3. \tag{10}$$

If $i_2 > 0$, then already for $i_2 = 1$, the left-hand side of (1) is greater than or equal to 3, so in this case, we must have $i_2 = 1$ and $i_0 = i_1 = 0$. This leads to the term $d \cdot \dfrac{\partial^2 u}{\partial x^2}$ for some d.

Let us consider the remaining case $i_2 = 0$. In this case, the equation (10) has the form $i_0 + 2 \cdot i_1 = 3$. Since $i_0 \geq 0$, we have $2i_1 \leq 3$, so we have two options: $i_1 = 0$ and $i_1 = 1$.

- For $i_1 = 0$, we have $i_0 = 3$, so we get a term proportional to u^3.
- For $i_1 = 1$, we get $i_0 = 1$, so we get a term proportional to $u \cdot \dfrac{\partial u}{\partial x}$.

Thus, we arrive at the following conclusion.

Conclusion: which equations have the desired symmetry. We have shown that any equation invariant under the desired symmetry has the form

$$\frac{\partial u}{\partial t} = a \cdot u \cdot \frac{\partial u}{\partial x} + d \cdot \frac{\partial^2 u}{\partial x^2} + b \cdot u^3. \tag{11}$$

By changing the unit of x to $|a|$ times smaller one (and maybe changing the direction of x), we can make the coefficient a to be equal to -1:

$$\frac{\partial u}{\partial t} = -u \cdot \frac{\partial u}{\partial x} + d \cdot \frac{\partial^2 u}{\partial x^2} + b \cdot u^3. \tag{12}$$

This is *almost* the Burgers' equation, the only difference is the new term $b \cdot u^3$.

This term can be excluded is we take an additional assumption that if the situation is spatially homogeneous, i.e., if $\dfrac{\partial u}{\partial x} \equiv 0$, then there is no change in time, i.e., $\dfrac{\partial u}{\partial t} = 0$.

How can we justify this additional requirement? Suppose that this requirement is not satisfied; then, in the homogeneous case, we have

$$\frac{du}{dt} = b \cdot u^3,$$

i.e., equivalently,

$$\frac{du}{u^3} = b \cdot dt$$

and, after integration, $u^{-2} = A \cdot t + B$ for some A and B. Thus, we have

$$u = \frac{1}{\sqrt{A \cdot t + B}}.$$

If we want to avoid such a spontaneous increase or decrease, then, from the invariance requirement, we get only the Burgers' equation.

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