Cooperativity-Preserving Observer Synthesis for the Control of Linear Continuous-Time Systems with Interval Uncertainty

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Abstract. Cooperative dynamic system models are characterized by state trajectories which are monotonic with respect to the systems' initial conditions. If this property is satisfied, it is possible to compute worst-case bounds for those states that are reachable within a finite time horizon by evaluating two independent initial value problems, one for the lower and one for the upper state boundaries. Preserving and ensuring this property also in cases in which state observers are used to estimate non-measurable state variables in real time is of special interest. If this property is satisfied, a computationally efficient verified proof becomes possible which allows for determining lower and upper bounds of each state variable and for checking whether state constraints are guaranteed to be obeyed if the aforementioned state estimates are included in a feedback control approach. The verification then has to show that the set of all possible state trajectories is guaranteed to be bounded by the minimum and/or maximum admissible values for selected components of the state vector. This contribution presents suitable observer-based control parameterizations relying on linear matrix inequality techniques for dynamic systems with interval uncertainty that can be used to parameterize and optimize state observers and eventually to preserve cooperativity of the closed-loop control system.

Keywords: State constraints · Cooperativity · Linear Dynamic Systems · Interval Analysis · Linear Matrix Inequalities.

1 Cooperativity of Continuous-Time Dynamic Systems

1.1 Verification of the System Property of Cooperativity

A sufficient criterion for cooperativity [14] of a general nonlinear autonomous dynamic system

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t)\right) \;, \quad \mathbf{x} \in \mathbb{R}^n \tag{1}$$

is that all off-diagonal elements $J_{i,j}$, $i, j \in \{1, \ldots, n\}$, $i \neq j$, of the corresponding Jacobian

$$\mathbf{J} = \frac{\partial \mathbf{f}\left(\mathbf{x}\right)}{\partial \mathbf{x}} \tag{2}$$

are strictly non-negative: $J_{i,j} \geq 0$, $i,j \in \{1,\dots,n\}$, $i \neq j$. For such cases, it is guaranteed that state trajectories $\mathbf{x}(t)$ starting in the positive orthant

$$\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \geq 0 \quad \forall i \in \{1,\dots,n\}\} \tag{3}$$

stay in this positive orthant for all points of time $t \geq 0$. This is due to the fact that the inequality

$$\dot{x}_i(t) = f_i\left(x_1,\dots,x_{i-1},0,x_{i+1},\dots x_n\right) \geq 0 \tag{4}$$

holds for each component $i \in \{1,\dots,n\}$ of the state vector as soon as the state x_i reaches the value $x_i = 0$. The property that all state variables of the above-mentioned system remain non-negative for all points of time is often referred to as *positivity* of the system model [5]. For such positive systems, the computation of worst-case enclosures of the system states with uncertain initial conditions can be simplified to a large extent. Especially, the use of general-purpose, set-valued solvers for initial value problems for ordinary differential equations, see [8, 9], can be avoided in this case, if initial states in the n-dimensional interval box

$$\mathbf{x}(0) \in \ [\mathbf{x}_0] = [\mathbf{x}]\,(0) = \begin{bmatrix} [\underline{x}_1(0)\ ;\ \overline{x}_1(0)] \\ \vdots \\ [\underline{x}_n(0)\ ;\ \overline{x}_n(0)] \end{bmatrix} \tag{5}$$

are taken into consideration with $\underline{\mathbf{x}}_0 \in \mathbb{R}^n_+$, where $\inf\left([x_i]\right) = \underline{x}_i$ represents the infimum and $\sup\left([x_i]\right) = \overline{x}_i$ the supremum of each vector component $[x_i] = [\underline{x}_i\ ;\ \overline{x}_i]$, $\underline{x}_i \leq x_i \leq \overline{x}_i$, $i \in \{1,\dots,n\}$.

It is then sufficient to derive two independent bounding systems in the form

$$\mathbf{f}_v\left(\mathbf{v}(t)\right) \leq \dot{\mathbf{x}}(t) \leq \mathbf{f}_w\left(\mathbf{w}(t)\right), \tag{6}$$

in which the component-wise defined inequalities on the variation rates of the state vector $\mathbf{x}(t)$ in (6) allow for computing state enclosures in the form $\mathbf{x}(t) \in [\mathbf{v}(t)\ ;\ \mathbf{w}(t)]$ that are valid for all $t \geq 0$.

1.2 Cooperative System Models in Various Fields of Applications

Cooperative system models that arise directly from a mathematical derivation of state equations using first-principle modeling approaches are typical for numerous applications in physics, biology, and (bio-)chemistry, such as

– mass transfer processes in fluidic compartment models [2],
– energy transfer between finite volume elements in thermal systems [10],
– compartment models of cell maturation processes, where the weighting factors at the graph edges represent the growth of bacteria or other microorganisms according to the Monod kinetics [4],
– enzymatic reactions according to the Michaelis-Menten kinetics, see [1],
– pressure dynamics in hydraulic and pneumatic networks with lumped storage elements, etc.

If system models are not naturally cooperative, several procedures for similarity transformations were recently developed which provide powerful techniques, especially in the case of linear dynamics, to transform the derived system model into an equivalent cooperative state-space representation [6].

2 Cooperativity-Preserving Observer-Based Feedback Control

For many dynamical systems from the application areas mentioned in the previous subsection, an interval-based parameter identification according to [10] leads to a description of possible parameterizations of the state equations that are described by the union of multiple subdomains. If — for each of these parameter subdomains with the associated interval vectors $\mathbf{p}_i \in \left[\underline{\mathbf{p}}_i \ ; \ \overline{\mathbf{p}}_i\right]$, $i \in \{1, \ldots, L\}$ — linear state equations with parameter-dependent system representations

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}_i \mathbf{x}_i(t) + \mathbf{B}_i \mathbf{u}(t) \ , \quad \mathbf{A}_i \in \left[\underline{\mathbf{A}}_i \ ; \ \overline{\mathbf{A}}_i\right] \ , \quad \mathbf{B}_i \in \left[\underline{\mathbf{B}}_i \ ; \ \overline{\mathbf{B}}_i\right] \tag{7}$$

and element-wise non-negative input matrices $\mathbf{B}_i$ with $\mathbf{u} \in \mathbb{R}_+^m$ can be found, a bank of L interval observers can be implemented which allows for estimating guaranteed lower and upper bounds of all states $\mathbf{x}_i(t) \in [\mathbf{v}_i(t) \ ; \ \mathbf{w}_i(t)]$ that are compatible with the models (7) and corresponding measurements

$$\mathbf{y}_{\mathrm{m}}(t) \in [\mathbf{y}_{\mathrm{m}}](t) = \mathbf{C}\mathbf{x}(t) + [-\Delta\mathbf{y}_{\mathrm{m}} \ ; \ \Delta\mathbf{y}_{\mathrm{m}}] \, . \tag{8}$$

To simplify the estimation of the sets of reachable states

$$\mathbf{x}(t) \in \bigcup_{i=1}^{L} [\mathbf{v}_i(t) \ ; \ \mathbf{w}_i(t)] \tag{9}$$

that are compatible with both (7) and (8) for all $i \in \{1, \ldots, L\}$, the observer design[1] should be performed in such a way that

1. the property of cooperativity remains satisfied despite the introduction of a measurement feedback in the observer differential equations

$$\left(\underline{\mathbf{A}}_i - \mathbf{H}\mathbf{C}\right)\hat{\mathbf{v}}_i + \underline{\mathbf{B}}_i\mathbf{u} + \mathbf{H}\underline{\mathbf{y}}_{\mathrm{m}} \leq \dot{\hat{\mathbf{x}}} \leq \left(\overline{\mathbf{A}}_i - \mathbf{H}\mathbf{C}\right)\hat{\mathbf{w}}_i + \overline{\mathbf{B}}_i\mathbf{u} + \mathbf{H}\overline{\mathbf{y}}_{\mathrm{m}} \tag{10}$$

   by the a-priori unknown matrix $\mathbf{H}$ and
2. the width $\hat{\mathbf{w}}_i - \hat{\mathbf{v}}_i$ of the estimated state enclosures is minimized while simultaneously accounting for the minimization of sensitivities of the estimated state enclosures against bounded measurement errors.

Using the estimates $\hat{\mathbf{v}}_i$ and $\hat{\mathbf{w}}_i$ for the lower and upper state bounds, it is then possible to implement closed-loop controllers in the form

$$\mathbf{u} = -\sum_{i=1}^{L} \left(\underline{\mathbf{K}}_i \hat{\mathbf{v}}_i + \overline{\mathbf{K}}_i \hat{\mathbf{w}}_i\right) \tag{11}$$

[1] Time arguments are subsequently omitted for brevity.

which can be seen as a natural extension of classical linear feedback control approaches [3] stabilizing the desired operating point $\mathbf{x} = \mathbf{0}$ in a guaranteed way despite bounded uncertainty in the system model (7).

As for the observer design, where it is necessary to determine the gain matrix $\mathbf{H}$, the choice of the controller gains $\underline{\mathbf{K}}_i$ and $\overline{\mathbf{K}}_i$ does not only have to account for stability of the system dynamics despite the before-mentioned uncertain parameters $\mathbf{p}_i$. In addition, the gains should be chosen in such a way that the trajectories become as insensitive as possible against the parameter uncertainty. On the one hand, this leads to the possibility to reduce the width of the domains of all reachable states by means of a feedback control. On the other hand, it is also possible to optimize the efficiency of the state observation approach (10) by reducing the widths of the intervals $[\hat{\mathbf{v}}_i \; ; \; \hat{\mathbf{w}}_i]$ if the control signal $\mathbf{u}$ is appropriately chosen.

This contribution gives an overview of computationally efficient approaches for the optimization of the observer parameterization (choice of the gain matrix $\mathbf{H}$) and the controller parameterization (choice of $\underline{\mathbf{K}}_i$ and $\overline{\mathbf{K}}_i$, respectively) so that the before-mentioned properties of cooperativity (allowing for a simplified computation of worst-case state enclosures), robust asymptotic stability (as an indispensable requirement for the applicability of any observer and controller), and optimality with respect to the tightness of the previously mentioned interval widths becomes possible. Novel computational techniques are derived and implemented with the help of solvers for linear matrix inequalities, see for example [7, 13, 15].

3 Applications and Outlook on Future Work

After a thorough review of the methodological background of the prerequisite for the solution of the before-mentioned tasks of observer and control parameterization, under the restriction to cooperativity of the resulting system dynamics, an overview of various applications from the field of control engineering is given. These applications are then also used to identify open problems with respect to the optimal parameterization of cooperativity-preserving state observers and/or controller synthesis as an outlook on future research.

Verified simulation results (for the computation of worst-case enclosures of reachable states with the goals of safety and feasibility analysis) as well as control and state estimation will be presented for

- distributed heating systems after a spatial semi-discretization [10],
- control and state estimation for high-temperature fuel cell stacks [11], and
- flexible high-speed rack feeder systems [12].

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